# SPMM800: Master's thesis in Mathematics
# - The Künneth Theorem for $C^*$-algebras -

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**Abstract (English):**
This thesis investigates the Künneth theorem in $K$-theory for $C^*$-algebras. We characterize a class of $C^*$-algebras for which the Künneth theorem holds and give a complete description of the associated Künneth short exact sequence. In particular, we analyze the classical Künneth map and provide a concrete description of the second homomorphism in the sequence, which, to our knowledge, has not previously appeared in the literature. Our approach uses $K$-theory with coefficients, geometric realizations, and free resolutions.

**Abstract (Danish):**
Denne afhandling undersøger Künneth-sætningen i $K$-teori for $C^*$-algebraer. Vi karakteriserer en klasse af $C^*$-algebraer, for hvilke Künneth sætningen gælder, og giver en fuldstændig beskrivelse af den tilhørende Künneth korte eksakte følge. Vi analyserer den klassiske Künneth afbildning og giver en konkret beskrivelse af den anden homomorfi i følgen, som efter vores bedste viden ikke tidligere har optrådt i litteraturen. Vores fremgangsmåde anvender $K$-teori med koefficienter, geometriske realisationer og frie resolutioner.

## Contents



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## 1 Introduction

Let $A$ and $B$ be $C^*$-algebras. There is a specific set of homomorphisms $K_q(A) \otimes K_p(B) \to K_{q+p}(A \otimes B)$ for $p, q \in \mathbb{Z}/2$ which relates the abelian group tensor product to the $C^*$-algebra tensor product in $K$-theory. The map $\alpha := \alpha_0 \oplus \alpha_1$ with

$$\alpha_0 : \begin{smallmatrix} K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B) \end{smallmatrix} \to K_0(A \otimes B) \qquad \alpha_1 : \begin{smallmatrix} K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B) \end{smallmatrix} \to K_1(A \otimes B),$$

incorporating these homomorphisms, is called the Künneth map. A related problem is to determine when the Künneth map is an isomorphism. The problem was first studied by [Atiyah] in the abelian case and subsequently by [Schochet]. The following is the original statement of Schochet.

**Theorem 1.1.** *[Schochet] Let $A$ and $B$ be $C^*$-algebras with $A$ in the smallest subcategory of the category of separable nuclear $C^*$-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\mathbb{Z}$ and by $\mathbb{R}$. Then there is a natural $\mathbb{Z}/2$-graded Künneth exact sequence*[1]

$$0 \longrightarrow \begin{smallmatrix} K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B) \end{smallmatrix} \xrightarrow{\alpha_0} K_0(A \otimes B) \xrightarrow{\beta_0} \begin{smallmatrix} \mathrm{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_0(B)) \end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix} K_0(A)\otimes K_1(B) \\ \oplus \\ K_1(A)\otimes K_0(B) \end{smallmatrix} \xrightarrow{\alpha_1} K_1(A \otimes B) \xrightarrow{\beta_1} \begin{smallmatrix} \mathrm{Tor}(K_0(A),K_0(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_1(B)) \end{smallmatrix} \longrightarrow 0$$

[1] Schochet uses notation to express both sequences as one, this notation will be introduced later in the text.

This theorem guarantees that a certain class of $C^*$-algebras has a Künneth short exact sequence with every other $C^*$-algebra. While the Künneth map $\alpha$ has a well known description, the homomorphism $\beta = \beta_0 \oplus \beta_1$ is without one. This does not come as a surprise since $\alpha$ was the original object of study, whereas $\beta$ has since appeared as a means to an end, it serves to measure the failure of $\alpha$ to be an isomorphism due to torsion. *The goal of this text is to obtain a concrete description of the map $\beta$, thereby achieving a complete description of the Künneth exact sequence.* To achieve generality, we will broaden our attention to the class of $C^*$-algebras $A$ which satisfies the Künneth short exact sequence for all $B$. This will mean cases of possibly non-nuclear $A$ and $B$, and thus we have to specify our tensor product. In our case we will look at the minimal and maximal tensor product.

Chapter 2 presents the self-contained, abelian group theoretic portion of the text. We begin by giving a definition of the torsion product in terms of generators, and we state several lemmas that will be used in the proof of Proposition 2.5. This proposition provides the machinery to construct the Künneth short exact sequence. In chapter 3, we introduce the notion of nuclearity and discuss several properties of nuclear $C^*$-algebras, including the behavior of tensorising with $C_0(X)$ for a locally compact Hausdorff space $X$. The central construction in this chapter is the mapping cone $C^*$-algebra of a $*$-homomorphism, which plays a key role in the subsequent chapters. Proposition 3.8 and Proposition 3.10 are both important and directly applicable to mapping cones. Chapter 4 begins with several $K$-theoretic lemmas, followed by a projection level motivation and definition of the Künneth map. We then establish equivalent characterizations of a $C^*$-algebra $A$ belonging to the class of $C^*$-algebras that satisfy the Künneth short exact sequence for all $C^*$-algebras $B$, for both the minimal and maximal tensor product. Finally, we define this class of $C^*$-algebras and prove some of its fundamental properties. Lemma 4.2 is very important and provides the $*$-homomorphisms we want to use in our mapping cone constructions. Chapter 5 is largely expository and is devoted to the definition of $K$-theory with coefficients. We study the Bockstein operations and examine their interactions in a big diagram. Chapter 6 brings together the results of the preceding chapters to obtain our description of the map $\beta$. The chapter follows a largely linear progression. We begin by defining the map $\Theta$. Lemma 6.4 establishes the interactions between $\Theta$ and the Bockstein operations introduced in Chapter 5. This interaction allows us to define a map $\theta$, which will ultimately be shown to be an isomorphism with its inverse implicitly being $\beta$. The large and technical Lemma 6.8 is required before we can finally prove, in Theorem 6.9, that $\theta$ is indeed the implicit inverse of $\beta$.

Throughout this thesis, $C^*$-algebras are denoted by $A, B, C$ and $D$. We write $\otimes_{\min}$ and $\otimes_{\max}$ for the maximal and minimal tensor products, respectively, and simply $\otimes$ when the tensor product is unique (for example, in the nuclear case) or when the tensor norm is clear from the context. To avoid cluttering diagrams, we write $M_n$ for $M_n(\mathbb{C})$.

## 2 The Torsion Group

$K$-theory innately deals with abelian groups, and in this chapter, we look at the abelian group theory that will support our efforts in the latter chapters. This chapter reviews a lot of chapter 8.2 and chapter 4 in [Fuchs]. We focus on the torsion product of abelian groups and the long exact sequence that connects them. Some of the results and definitions are only used internally to this chapter, an example is the following definition.

**Definition 2.1** (Direct limits)**.** Let $I$ be an index set that is partially ordered and has the property that for all $i, j \in I$ there exists $k \in I$ with $i \leq k$, $j \leq k$. Let $\{A_i\}_{i \in I}$ be a system of abelian groups such that for each indices with $i \leq j$ we have connecting homomorphisms $\pi_i^j : A_i \to A_j$ subject to

1. $\pi_i^i = \mathrm{id}_{A_i}$ for all $i \in I$
2. If $i \leq j \leq k$ then $\pi_j^k \pi_i^j = \pi_i^k$

We call $\mathfrak{U} = \langle A_i, \pi_i^j \rangle$ a direct system and by the direct limit of $\mathfrak{U}$ we mean an abelian group $A_*$ such that

1. There are maps $\pi_i : A_i \to A_*$ (for all $i \in I$) such that $\pi_i = \pi_j \pi_i^j$ for all $i \leq j$.
2. If $G$ is any abelian group with $\rho_i : A_i \to G$ (for all $i \in I$) satisfying $\rho_i = \rho_j \pi_i^j$ for all $i \leq j$, then there is a unique map $\alpha : A_* \to G$ such that $\rho_i = \alpha \pi_i$ for all $i \in I$.

Direct limits always exist and we denote these $A_* = \varinjlim_{i \in I} A_i$ and call $\{\pi_i : A_i \to A_*\}_{i \in I}$ the canonical homomorphisms.

There are two (equivalent) ways of defining the torsion product. One takes a low level perspective, which means working with generators. The other takes a high level approach, defining the torsion product as a kernel of a certain free resolution. This is the low level way of defining the torsion product.

**Definition 2.2** (Torsion Product)**.** The torsion product of abelian groups $G, H$ is again an abelian group $\mathrm{Tor}(G, H)$ defined on the generators:

$$(g, n, h) \text{ satisfying } ng = 0 = nh, \quad g \in G, h \in H, n \in \mathbb{N}$$

subject to the relations

$$\begin{array}{lrcll} 1. & (g,n,h_1+h_2) & = & (g,n,h_1)+(g,n,h_2) & (g\in G[n], h_1,h_2\in H[n]) \\ 2. & (g_1+g_2,n,h) & = & (g_1,n,h)+(g_2,n,h) & (g_1,g_2\in G[n], h\in H[n]) \\ 3. & (g,nm,h) & = & (mg,n,h) & (g\in G[nm], h\in H[n]) \\ 4. & (g,nm,h) & = & (g,n,mh) & (g\in G[n], h\in H[nm]) \end{array}$$

If we have $\alpha : G \to G'$ and $\gamma : H \to H'$, then can define a homomorphism

$$\mathrm{Tor}(\alpha,\gamma) : \mathrm{Tor}(G,H) \to \mathrm{Tor}(G',H')$$

by mapping the generators $(g,m,h) \mapsto (\alpha(g),m,\gamma(h))$. If we fix a group $H$, then we see that $\mathrm{Tor}(\alpha,H) : \mathrm{Tor}(G,H) \to \mathrm{Tor}(G',H)$ is additive in that

$$\begin{aligned}\mathrm{Tor}(\alpha_1+\alpha_2,H)(g,m,h) = (\alpha_1(g)+\alpha_2(g),m,h) = (\alpha_1(g),m,h)+(\alpha_2(g),m,h) \\ = \mathrm{Tor}(\alpha_1,H)(g,m,h)+\mathrm{Tor}(\alpha_2,H)(g,m,h)\end{aligned}$$

It is quite obvious that $\mathrm{Tor}(G,H) \cong \mathrm{Tor}(H,G)$ since this isomorphism is clear when taking a generator point of view. This means we only need to illustrate the following in the one argument case.

**Lemma 2.3.** Tor *commutes with direct limits (and direct sums, see remark below), that is*

$$\varinjlim_{i\in I} \mathrm{Tor}(A_i,C) = \mathrm{Tor}(\varinjlim_{i\in I} A_i, C)$$

*for all direct systems* $\mathfrak{A} = \langle A_i, \pi_i^j\rangle$ *and all abelian groups* $C$.

**Remark 2.4** (Tor commutes with direct sums)**.** If $\bigoplus_{i\in I} A_i$ is a direct sum over $I$, then a well-ordering $\leq$ of $I$ always exists. Now the direct sum can be represented as the direct limit

$$\bigoplus_{i\in A} A_i = \varinjlim_{i\in I} \bigoplus_{k\in I, k\leq i} A_k$$

of the direct system $\langle \bigoplus_{k\in I,k\leq i} A_k, \pi_i^j\rangle_{i\leq j}$ where for $i\leq j$ the homomorphisms $\pi_i^j : \bigoplus_{k\in I,k\leq i} A_k \to \bigoplus_{k\in I,k\leq j} A_k$ are the canonical inclusions. The Tor functor commutes with finite direct sums in the obvious way, this can be extended to the more general case by using Lemma 2.3 and the direct limit construction above

$$\begin{aligned}\bigoplus_{i\in I} \mathrm{Tor}(A_i,C) = \varinjlim_{i\in I} \bigoplus_{k\in I,k\leq i} \mathrm{Tor}(A_k,C) = \varinjlim_{i\in I} \mathrm{Tor}\left(\bigoplus_{k\in I,k\leq i} A_k, C\right) \\ = \mathrm{Tor}\left(\varinjlim_{i\in I} \bigoplus_{k\in I,k\leq i} A_k, C\right) = \mathrm{Tor}\left(\bigoplus_{i\in I} A_i, C\right)\end{aligned}$$

*Proof.* Let $\mathfrak{U} = \langle A_i, \pi_i^j \rangle$ be a direct system of abelian groups, denote the direct limit $A$ and the canonical maps $\pi_i : A_i \to A$ for all $i \in I$. We get a direct system

$$\langle \mathrm{Tor}(A_i, C), \mathrm{Tor}(\pi_i^j, \mathrm{id}_C) \rangle$$

since $\mathrm{Tor}(\pi_i^i, \mathrm{id}_C) = \mathrm{Tor}(\mathrm{id}_{A_i}, \mathrm{id}_C)$ for all $i \in I$ and $\mathrm{Tor}(\pi_i^k, \mathrm{id}_C) = \mathrm{Tor}(\pi_j^k \pi_i^j, \mathrm{id}_C) = \mathrm{Tor}(\pi_j^k, \mathrm{id}_C)\mathrm{Tor}(\pi_i^j, \mathrm{id}_C)$ for all $i \leq j \leq k$ in $I$. We know that a direct limit $T := \varinjlim_{i \in I} \mathrm{Tor}(A_i, C)$ exists along with canonical homomorphisms $\rho_i : \mathrm{Tor}(A_i, C) \to T$. The homomorphisms $\mathrm{Tor}(\pi_i, \mathrm{id}_C) : \mathrm{Tor}(A_i, C) \to \mathrm{Tor}(A, C)$ satisfy $\mathrm{Tor}(\pi_i, \mathrm{id}_C) = \mathrm{Tor}(\pi_j, \mathrm{id}_C)\mathrm{Tor}(\pi_i^j, \mathrm{id}_C)$ for $i \leq j$ and since $T$ is a direct limit, then there is a unique homomorphism $\sigma : T \to \mathrm{Tor}(A, C)$ with $\sigma\rho_i = \mathrm{Tor}(\pi_i, \mathrm{id}_C)$. We now need to show that $\sigma$ is an isomorphism.
Let $(a, m, c) \in \mathrm{Tor}(A, C)$, we can write $a = \pi_i a_i$ for some $a_i \in A_i$ with $ma_i = 0$. So $\mathrm{Tor}(\pi_i, \mathrm{id}_\mathrm{C})(a_i, m, c) = (a, m, c)$ and this shows that $\sigma$ is surjective. Now for injectivity, if $x \in \ker(\sigma)$, then there exists an index $i \in I$ such that $\rho_i y = x$ for some $y_i \in \mathrm{Tor}(A_i, C)$. Now we have $\mathrm{Tor}(\pi_i, \mathrm{id}_C)(y) = \sigma\rho_i(y) = \sigma(x) = 0$, this must mean that there exists $j \in I$ with $i \leq j$ such that $\mathrm{Tor}(\pi_i^j, \mathrm{id}_C)(y) = 0$. Pr. definition, we have that $\rho_j \mathrm{Tor}(\pi_i^j, \mathrm{id}_C) = \rho_i$, and so we have $x = \rho_i y = 0$. Now we conclude, since $\sigma$ is bijective, that $\sigma$ is an isomorphism. □

We will now introduce the first example of a relationship between the torsion product and the tensor product of abelian groups. This materializes as a connecting homomorphism $\delta$ on a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

We define $\delta : \mathrm{Tor}(G, C) \to G \otimes A$ as follows. If $(g, m, c) \in \mathrm{Tor}(G, C)$, then we have $\beta(b) = c$ for some $b \in B$. Let $\delta : (g, m, c) \mapsto g \otimes a$ for some $a \in A$ with $\alpha(a) = mb$, this $a$ exists because the sequence is exact and $\beta(mb) = 0$. Since $\delta$ is defined on all generators of Tor, then it extends to all of $\mathrm{Tor}(G, C)$. $\delta$ is independent of the choice of $a$ and $b$ since if $a'$, $b'$ are other choices, then $b' = b + \alpha(x)$, $a' = a + mx$ for some $x \in A$. This implies that $g \otimes a' = g \otimes a + g \otimes mx$ where $g \otimes mx = mg \otimes x = 0$, thus the choice we make about $a$ and $b$ is irrelevant to the definition of $\delta$. It is clear that the map is defined on the generators, but we still need to check that the relations on the generators of $\mathrm{Tor}(G, C)$ are respected by the map $\delta$, only then will $\delta$ be a homomorphism. We will check the four relations of Definition 2.2.

1. $\delta(g, m, c_1 + c_2) = \delta(g, m, c_1) + \delta(g, m, c_2)$ for $g \in G[m]$, $c_1, c_2 \in H[m]$:
Lift $c_1, c_2 \in C$ to $b_1, b_2 \in B$ respectively, then we have unique $a_1, a_2 \in A$ such that $\alpha(a_1) = mb_1$, $\alpha(a_2) = mb_2$. Additionally, we have that $c_1 + c_2 \in C$ lifts to

$b_1 + b_2 \in B$ with $\alpha(a_1 + a_2) = m(b_1 + b_2)$. This means that $\delta(g, m, c_1 + c_2) = g \otimes (a_1 + a_1) = g \otimes a_1 + g \otimes a_2 = \delta(g, m, c_1) + \delta(g, m, c_2)$.

2. $\delta(g_1 + g_2, m, c) = \delta(g_1, m, c) + \delta(g_2, m, c)$ for $g_1, g_2 \in G[m]$, $c \in H[m]$:
$\delta(g_1 + g_2, m, c) = (g_1 + g_2) \otimes a = g_1 \otimes a + g_2 \otimes a = \delta(g_1, m, c) + \delta(g_2, m, c)$ where $a$ is picked according to the rules.

3. $\delta(ng, m, c) = \delta(g, mn, c)$ for $g \in G[mn]$, $c \in H[m]$:
Let $c \in C$ lift to $b \in B$, then $\beta(mb) = 0$ and so there exists $a \in A$ with $\alpha(a) = mb$. We also have $\beta(mnb) = 0$ and so there exists $a' \in A$ with $\alpha(a') = mnb$, but we see that $\alpha(na) = \alpha(a')$ so $na = a'$. So we have $\delta(ng, m, c) = ng \otimes a = g \otimes na = g \otimes a' = \delta(g, mn, c)$

4. $\delta(g, m, nc) = \delta(g, mn, c)$ for $g \in G[m]$, $c \in H[mn]$:
Let $b \in B$ be a lift of $c$, then $nb \in B$ is a lift of $nc$. Then $\beta(m(nb)) = \beta((nm)b) = 0$ and so the this element lifts to $a \in A$ with $\alpha(a) = nmb$. So we have $\delta(g, m, nc) = g \otimes a = \delta(g, mn, c)$.

This proves that $\delta$ is a homomorphism.

This connecting homomorphism is used in the following proposition.

**Proposition 2.5.** *For any exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of abelian groups, and for any abelian group $G$, the following sequence is exact*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(G,A) \arrow[r, "\alpha_*"] & \mathrm{Tor}(G,B) \arrow[r, "\beta_*"] & \mathrm{Tor}(G,C) \arrow[dll, rounded corners, "\delta"] & \\
 & G \otimes A \arrow[r, "\mathrm{id}_G \otimes \alpha"] & G \otimes B \arrow[r, "\mathrm{id}_G \otimes \beta"] & G \otimes C \arrow[r] & 0
\end{tikzcd}$$

*where $\delta$ is the connecting homomorphism.*

*Proof.* Let $I$ be a directed set and let $\langle A_i, f_i^j \rangle, \langle B_i, g_i^j \rangle, \langle C_i, h_i^j \rangle$ be direct systems with respective direct limits $A, B, C$. Let

$$0 \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \longrightarrow 0 \tag{$\star$}$$

be an exact short sequence for each $i \in I$ with $\beta_j(g_i^j(b_i)) = h_i^j(\beta_i(b_i))$ and $\alpha_j(f_i^j(a_i)) = g_i^j(\alpha_i(a_i))$ for all $i \le j$ in $I$. We want to show that the direct limit of this sequence is again a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. We define the limit of the homomorphisms $\alpha_i$ as $\alpha : A \to B$ in the following way, for each $a \in A$ we have $i \in I$ and $a_i \in A_i$ with $f_i(a_i) = a$, let $\alpha(a) = g_i \alpha_i(a_i)$ for one such $a_i \in A_i$. This definition is independent of the choice of $i \in I$ and $a_i \in A_i$. The definition of $\beta : B \to C$ uses the same procedure.

As a start, we can show that $\beta\alpha(a) = 0$ for all $a \in A$. This is apparent since $\beta\alpha(a) = \beta g_i\alpha_i(a_i) = h_i\beta_i\alpha_i(a_i) = h_i(0) = 0$ for all $i \in I$ and $a_i \in A_i$ with the property $f_i(a_i) = a$. We have $\mathrm{im}(\alpha) \subseteq \ker(\beta)$ and for the opposite direction we shall assume that $b \in B$ with $\beta(b) = 0$. We know there exists $i \in I$ and $b_i \in B_i$ with $g_i(b_i) = b$ and that there exists $j \geq i$ such that $h_i^j(\beta_i(b_i)) = 0$. The maps commute in the following way $0 = h_i^j(\beta_i(b_i)) = \beta_j(g_i^j(b_i))$ and from exactness we have an $a_j \in A_j$ with $\alpha_j(a_j) = g_i^j(b_i)$. This means that $\alpha(f_j(a_j)) = g_j\alpha_j(a_j) = g_jg_i^j(b_i) = g_i(b_i) = b$ and finally we have $\mathrm{im}(\alpha) = \ker(\beta)$.
For surjectivity of $\beta$, let $c \in C$ and consequently $c_i \in C_i$ exists with $h_i(c_i) = c$. This means that $\beta_i(b_i) = c_i$ for some $b_i \in B_i$. Now we have $\beta(g_i(b_i)) = h_i\beta_i(b_i) = h_i(c_i) = c$. For injectivity, let $\alpha(a) = 0$ for $a \in A$, then $g_i\alpha_i(a_i) = 0$ for some $a_i \in A_i$ with $f_i(a_i) = a$, going further we can find $g_i^j(\alpha_i(a_i)) = 0 \in B_j$ for some $j \geq i$ in $I$. Now $0 = g_i^j(\alpha_i(a_i)) = \alpha_j(f_i^j(a_i))$ and from injectivity we have $f_i^j(a_i) = 0$. This concludes with $a = f_i(a_i) = 0$.
Every abelian group is a direct limit of its finitely generated subgroups which are always of the form $\mathbb{Z}^n \oplus \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_u$, where $q_i | q_{i+1}$ for all $i = 1, ...u-1$. Both Tor and the tensor product commute with direct limits. If $\{G_i\}_{i\in I}$ is the finitely generated abelian groups in the direct system with direct limit $G$, then we see that exactness of

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{i\in I} \mathrm{Tor}(G_i, A) \arrow[r, "\alpha_*"] & \varinjlim_{i\in I} \mathrm{Tor}(G_i, B) \arrow[r, "\beta_*"] & \varinjlim_{i\in I} \mathrm{Tor}(G_i, C) \arrow[dll, rounded corners, "\delta_i"] & \\
& \varinjlim_{i\in I}(G_i \otimes A) \arrow[r, "\mathrm{id}_{G_i}\otimes\alpha"] & \varinjlim_{i\in I}(G_i \otimes B) \arrow[r, "\mathrm{id}_{G_i}\otimes\beta"] & \varinjlim_{i\in I}(G_i \otimes C) \arrow[r] & 0
\end{tikzcd}$$

which implies the Proposition 2.5 statement, boils down to proving that the Proposition 2.5 sequence is exact for $G = \mathbb{Z}$ and $G = \mathbb{Z}/m$ for $m \in \mathbb{N}$. Observe that the direct limit structure commutes with the exact sequence maps as explained in $\star$ in the beginning of the proof. Now we will handle the two different cases.
If $G = \mathbb{Z}$, then the sequence is trivial since the top row is 0 and the bottom is the original short exact sequence.
If $G = \langle g \rangle \cong \mathbb{Z}/m$ for some $m \in \mathbb{N}$. Then generators of the Tor groups are of the form $(ng, k, x)$ $(x \in X = A, B, C)$ with $n = 0, 1, ..., m-1$, $m|nk$ and $kx = 0$. So we have $(ng, k, x) = (g, cm, x)$ for some $c \in \mathbb{N}$ and in turn $(g, cm, x) = (g, m, cx)$. This means every element of Tor is of the form $(g, m, x)$ with $mx = 0$. Elementary tensors in the tensor products $G \otimes X$ $(X = A, B, C)$ will look like $ng \otimes x = g \otimes nx$ for $n = 0, 1, ..., m-1$ and $x \in X$. So elementary tensors are always of the form $g \otimes x$ where $x$ is taken mod $mX$. Now for exactness, we first need to make sure that the composition of any two maps is zero. It is clear that all compositions not involving

$\delta$ is zero. We need to check the two additional compositions

$$\delta\beta_*(g,m,b) = \delta(g,m,\beta(b)) = g \otimes a = 0 \text{ since } \alpha(a) = mb = 0 \text{ and } \alpha \text{ is injective}$$
$$(\mathrm{id}_G \otimes \alpha)\delta(g,m,c) = (\mathrm{id}_G \otimes \alpha)(g \otimes a) = g \otimes \alpha(a) = 0 \text{ as } \alpha(a) \in mB$$

We now need to prove that the kernels of the maps are contained in the images of the previous. The first 3 are the ones we need to check,
if $\alpha_*(g,m,a) = (g,m,\alpha(a)) = 0$, then $\alpha(a) = 0$ and thus $a = 0$, so $\alpha_*$ is injective. If $\beta_*(g,m,b) = 0$ then $\beta(b) = 0$ and exactness of the original sequence ensures that $a \in A$ with $\alpha(a) = b$ exists, we also have $ma = 0$. It is clear that $\alpha_*(g,m,a) = (g,m,b)$ and we have exactness at the second Tor group.
Now suppose that $\delta(g,m,c) = g \otimes a = 0$, this implies that $a \in mA$ and so $\alpha(ma') = \alpha(a) = mb$ for some $a' \in A$. Rearranging, we get $m(b - \alpha(a)) = 0$ and hence $(a,m,b-\alpha(a)) \in \mathrm{Tor}(G,B)$ is sent to $(g,m,c) \in \mathrm{Tor}(G,C)$. For exactness at $G \otimes A$, let $(\mathrm{id}_A \otimes \alpha)(g \otimes a) = 0$, then $\alpha(a) = mb$ for some $b \in B$ and we get $\delta(g,m,\beta(b)) = g \otimes a$. Exactness at $G \otimes B$ and $G \otimes C$ is automatic. □

Proposition 2.5 gives us new way to determine the torsion product of two abelian groups. Instead of the generator approach which starts from the bottom up, we can infer the structure of our torsion product in the following way.

**Definition 2.6** (Torsion product from a free resolution)**.** To compute $\mathrm{Tor}(A,B)$ where $A,B$ are abelian groups. Get a free resolution $F_1 \xrightarrow{\nu} F_2 \to B \to 0$, then tensorize with $A$ to get

$$A \otimes F_1 \xrightarrow{\mathrm{id}_A \otimes \nu} A \otimes F_2 \longrightarrow A \otimes B \longrightarrow 0$$

Now we have $\mathrm{Tor}(A,B) \cong \ker(\mathrm{id}_A \otimes \nu)$ and $A \otimes B \cong \mathrm{coker}(\mathrm{id}_A \otimes \nu)$

## 3 Nuclearity, Pullbacks and Mapping Cones

In this chapter we will introduce the notion of nuclearity and discuss several properties of nuclear $C^*$-algebras, including the behavior of tensorising with $C_0(X)$ for a locally compact Hausdorff space $X$. We directly reference [Blackadder] for the nuclearity part of this chapter and [Uuye] for inspiring the structure of this mapping cone exposition. The central construction in this chapter is the mapping cone $C^*$-algebra of a $*$-homomorphism. Proposition 3.8 and Proposition 3.10 are both important and directly applicable to mapping cones.

**Definition 3.1.** Let $A$ be a $C^*$-algebra. Then $A$ is said to be nuclear, if for every $C^*$-algebra $B$, the algebraic tensor product $A \odot B$ has a unique $C^*$-norm.

The reader familiar with $C^*$-norms on tensor products knows that $A \otimes_{\min} B \cong A \otimes_{\max} B$ does not hold in general. If a $C^*$-algebra is nuclear, then we write $\otimes$ instead of $\otimes_{\min}$ or $\otimes_{\max}$, which in this case is the same. For this special class of $C^*$-algebras, we have the following.

**Lemma 3.2.** *[Blackadder, Theorem 15.8.2] The class of nuclear $C^*$-algebras*

- *contain all commutative $C^*$-algebras.*
- *contain $M_n$ for all $n \in \mathbb{N}$ and $\mathcal{K}$ which is the compact operators on an infinite-dimensional separable Hilbert space.*
- *is closed under the 2-out-of-3 property: If a short exact sequence $0 \to J \to A \to A/J \to 0$ of $C^*$-algebras is such that two out of three $C^*$-algebras are nuclear, then the last one is also nuclear.*
- *is closed under tensor products.*

**Corollary 3.3.** *$C_0(X)$ is nuclear if $X$ is locally compact and Hausdorff.*

*Proof.* This is easily seen as $C_0(X)$ is commutative. □

**Lemma 3.4.** *$C_0(X, A) \cong C_0(X) \otimes_{\min} A \cong C_0(X) \otimes_{\max} A$ for all local compact Hausdorff spaces $X$ and $C^*$-algebras $A$.*

*Proof.* Corollary 3.3 states that $C_0(X)$ is nuclear, so we will prove $C_0(X, A) \cong C_0(X) \otimes A$ for the unique tensor product norm. Let us define our mapping $\varphi : C_0(X) \otimes A \to C_0(X, A)$ by the assignment $f \otimes a \mapsto (t \mapsto f(t)a)$. This is clearly a $*$-homomorphism.
For injectivity. Let $\varphi(f \otimes a) = \varphi(g \otimes b)$ for some $f, g \in C_0(X)$ and $a, b \in A$. If $a, b$ are linearly independent, then $f(t)a = g(t)b$, for all $t \in X$, implies either $f(t) = g(t) = 0$ for all $t \in X$ or $a, b = 0$. Either way we have $f \otimes a = g \otimes b$. Now if $a, b$ are linearly dependent, then $a = \lambda b$ for some $\lambda \in \mathbb{C}$, so we have $f(t)\lambda b = g(t)b$ and in turn $\lambda f(t) = g(t)$ for all $t \in X$. Now we see that $f \otimes a = f \otimes \lambda b = \lambda f \otimes b = g \otimes b$ which implies that $\varphi$ is injective. To see that $\varphi$ is surjective, it is enough to establish that $\operatorname{span}\{fa \mid f \in C_0(X),\ a \in A\}$ is dense in $C_0(X, A)$. We refer to [Rørdam, Lemma 10.1.1] for a proof of this claim. □

**Corollary 3.5.** *$C_0(X, A) \otimes_\alpha B \cong C_0(X, A \otimes_\alpha B)$, where $\alpha = \min/\max$, for all local compact Hausdorff spaces $X$ and $C^*$-algebras $A, B$.*

*Proof.* Associativity of the tensor product combined with Lemma 3.4 gives the following isomorphisms

$$C_0(X, A) \otimes_\alpha B \cong (C_0(X) \otimes_\alpha A) \otimes_\alpha B \cong C_0(X) \otimes_\alpha (A \otimes_\alpha B) \cong C_0(X, A \otimes_\alpha B)$$

□

Now we have come to the most important construction in this chapter. As a special case of $C_0(X)$ for a local compact Hausdorff space $X$, we have $C_0(0,1] = \{\text{continuous } f : [0,1] \to \mathbb{C} \mid f(0) = 0\}$ with the point evaluation $*$-homomorphism $\mathrm{ev}_1 : C_0(0,1] \to \mathbb{C}$ given by $f \mapsto f(1)$. Now we have the definition.

**Definition 3.6.** Let $\phi : A \to B$ be a $*$-homomorphism, the mapping cone $C_\phi$ associated with $\phi$ is the pullback

$$\begin{tikzcd}
C_\phi \arrow[r] \arrow[d] & C_0(0,1] \otimes B \arrow[d, "\mathrm{ev}_1 \otimes \mathrm{id}_B"] \\
A \arrow[r, "\phi"] & B
\end{tikzcd}$$

The reader may be familiar with the short exact sequence $0 \to SA \to CA \to A \to 0$ made from a $C^*$-algebra $A$ and its suspension and cone. We have the analogous short exact sequence incorporating the mapping cone.

$$0 \to SB \xrightarrow{\sigma} C_\phi \xrightarrow{\eta} A \to 0$$

where if we view $C_\phi := \{(a,f) \in A \oplus C_0((0,1], B) \mid \phi(a) = f(1)\}$ (made possible by Lemma 3.4), then the maps are $\sigma(f) = (0,f) \in C_\phi$ and $\eta(a,f) = a$. This exact sequence gives us the six-term exact sequence [Rørdam, Theorem 10.1.3, 11.1.2, 12.1.2]

$$\begin{tikzcd}
K_0(SB) \arrow[r, "\sigma_*"] & K_0(C_\phi) \arrow[r, "\eta_*"] & K_0(A) \arrow[d, "\partial_1"] \\
K_1(A) \arrow[u, "\partial_2"] & K_1(C_\phi) \arrow[l, "\eta_*"] & K_1(SB) \arrow[l, "\sigma_*"]
\end{tikzcd}$$

and since $K_1(SB) \cong K_0(B)$ and $K_0(SB) \cong K_1(B)$, then we have the six-term exact sequence

$$\begin{tikzcd}
K_0(C_\phi) \arrow[r, "\eta_*"] & K_0(A) \arrow[r, "\phi_*"] & K_0(B) \arrow[d, "\sigma_*"] \\
K_1(B) \arrow[u, "\sigma_*"] & K_1(A) \arrow[l, "\phi_*"] & K_1(C_\phi) \arrow[l, "\eta_*"]
\end{tikzcd}$$

and it turns out that $\phi_*$ is actually the boundary map. This is explained in the proof of *2.* $\Rightarrow$ *3.* in Theorem 4.5. Exactness of the six-term sequence gives us the following result.

**Corollary 3.7.** *Let $\phi : A \to B$ be a $*$-homomorphism. Then $\phi$ induces an isomorphism $\phi_* : K_*(A) \to K_*(B)$ if and only if $K_*(C_\phi) = 0$.*

The following proposition will be used heavily in the later chapters.

**Proposition 3.8.** *Consider the commutative diagram of $C^*$-algebras*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & X \arrow[r, "\alpha"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & I \arrow[r] & D \arrow[r, "\eta"] & B \arrow[r] & 0
\end{tikzcd}$$

*where the bottom row is exact. Then the top row is exact if and only if the right-hand square is a pullback.*

*Proof.* Assume that the top row, as well as the bottom row, is exact. If $Y$ is a $C^*$-algebra such that $\eta \circ f = \beta \circ g$, then we want to prove that there exists a unique $h$ making the following diagram commute.

$$\begin{tikzcd}
 & & Y \arrow[d, dashed, "h"] \arrow[dr, bend left, "g"] \arrow[d, bend right=60, "f"'] & & \\
0 \arrow[r] & I \arrow[r, "j"] \arrow[d, equal] & X \arrow[r, "\alpha"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & I \arrow[r, "i"] & D \arrow[r, "\eta"] & B \arrow[r] & 0
\end{tikzcd}$$

Let $y \in Y$, we define $h(y) \in X$ by the following procedure. Pick an arbitrary element $x \in \alpha^{-1}(g(y)) \neq \emptyset$, then since $\eta \circ f(y) = \eta \circ \sigma(x)$ we have $\eta(f(y) - \sigma(x)) = 0$ and so we have a unique $e \in I$ such that $i(e) = f(y) - \sigma(x)$, equivalently $f(y) = \sigma(x) + \sigma \circ j(e) = \sigma(x + j(e))$. This gives us a well-defined $h(y) = x + j(e)$ in this way. Now we have a commuting diagram, but we still need to ensure that $h$ is a $*$-homomorphism.

Let $y, z \in Y$ and $\lambda \in \mathbb{C}$, then we have

$$\sigma(h(y + \lambda z)) = f(y + \lambda z) = f(y) + \lambda f(z) = \sigma(h(y) + \lambda h(z))$$

and so $\sigma(h(y + \lambda z) - h(y) - \lambda h(z)) = 0$. If $h(y) = x_1 + j(e_1)$ and $h(z) = x_2 + j(e_2)$, then observe that $x_1 + \lambda x_2 \in \alpha^{-1}(g(y + \lambda z)) = \alpha^{-1}(g(y) + \lambda g(z))$, so we have $\alpha(h(y + \lambda z) - h(y) - \lambda h(z)) = 0$. This implies that there exists a $k \in I$ such that $j(k) = h(y + \lambda z) - h(y) - \lambda h(z)$. Since $\sigma \circ j = i$ we have

$$i(k) = \sigma(h(y + \lambda z) - h(y) - \lambda h(z)) = 0$$

But since $i$ is injective we know that $k = 0$, but this implies that $j(k) = h(y + \lambda z) - h(y) - \lambda h(z) = 0$. So $h(y + \lambda z) = h(y) + \lambda h(z)$. The exact same argument shows that, in addition to $h$ being linear, it also preserves multiplication and the involution. So $h$ is a $*$-homomorphism.

Now assume that $X$ is a pullback, this means that $X = \{(d,a) \in D \oplus A \mid \eta(d) = \beta(a)\}$ and $\sigma, \alpha$ are the projection maps. $\alpha$ is a surjection since $\eta$ is a surjection, indeed for all $a \in A$ we have $\beta(a) \in \operatorname{im}(\eta)$, so $(d,a) \in X$ for some $d \in D$ with $\alpha(d,a) = a$.
$j$ can not be non-injective since $i$ is injective and $i = \sigma \circ j$, so $i$ is injective. We now only need to prove that the image of $j$ is the kernel of $\alpha$. Observe that $\ker(\alpha) = \{(d,0) \in D \oplus A \mid \eta(d) = 0 = \beta(0)\} = \{(d,0) \mid d \in \ker(\eta)\} = \{(d,0) \mid d \in \operatorname{im}(i)\}$. Now if $(d,0) \in \ker(\alpha)$, we have $i(k) = d$ for some $k \in I$, this implies $j(k) = (d,0)$ and $\ker(\alpha) \subseteq \operatorname{im}(j)$. Let $k \in I$, then we have $j(k) = (i(k),0) \in \ker(\alpha)$ thus proving $\ker(\alpha) = \operatorname{im}(j)$. □

**Corollary 3.9.** *Let $P = A \oplus_C B$ be a pullback, then $P \otimes_{\max} D = (A \otimes_{\max} D) \oplus_{C \otimes_{\max} D} (B \otimes_{\max} D)$ where the relevant $*$-homomorphisms are tensored with the identity map $\mathrm{id}_D$ on $D$.*

*Proof.* Let $P$ be the mentioned pullback, then the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\alpha) \arrow[r, "\iota'"] \arrow[d, equal] & P \arrow[r, "\delta"] \arrow[d, "\sigma"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \ker(\alpha) \arrow[r, "\iota"] & A \arrow[r, "\alpha"] & C \arrow[r] & 0
\end{tikzcd}$$

with $\iota'(i) = (i,0) \in P$ for all $i \in \ker(\alpha)$ has an exact top row by Proposition 3.8. Since the maximal tensor product preserves exactness, then for any $C^*$-algebra $D$ we have the commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(\alpha) \otimes_{\max} D \arrow[r, "\iota' \otimes \mathrm{id}_D"] \arrow[d, equal] & P \otimes_{\max} D \arrow[r, "\delta \otimes \mathrm{id}_D"] \arrow[d, "\sigma \otimes \mathrm{id}_D"] & B \otimes_{\max} D \arrow[r] \arrow[d, "\beta \otimes \mathrm{id}_D"] & 0 \\
0 \arrow[r] & \ker(\alpha) \otimes_{\max} D \arrow[r, "\iota \otimes \mathrm{id}_D"] & A \otimes_{\max} D \arrow[r, "\alpha \otimes \mathrm{id}_D"] & C \otimes_{\max} D \arrow[r] & 0
\end{tikzcd}$$

and this shows by Proposition 3.8 that $P \otimes_{\max} D$ is the pullback from $\alpha \otimes \mathrm{id}_D$ and $\beta \otimes \mathrm{id}_D$. □

**Proposition 3.10.** *let $\phi : A \to B$ be a $*$-homomorphism and let $D$ be a $C^*$-algebra. Then we have natural isomorphisms*

$$C_\phi \otimes_{\max} D \cong C_{\phi \otimes_{\max} \mathrm{id}_D}$$
$$C_\phi \otimes_{\min} D \cong C_{\phi \otimes_{\min} \mathrm{id}_D}$$

*Proof.* Since both $C_0(0,1]$ and $\mathbb{C}$ are nuclear, then we have $\mathrm{ev}_1 \otimes \mathrm{id}_B : C_0(0,1] \otimes B \to \mathbb{C} \otimes B \cong B$. The definition of the mapping cone of $\phi$ is the pullback

$$\begin{tikzcd} C_\phi \arrow[r] \arrow[d] & C_0(0,1] \otimes B \arrow[d, "\mathrm{ev}_1 \otimes \mathrm{id}_B"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

Let us start of by proving the first part, keep in mind that the tensor product is associative. The definition of $C_{\phi \otimes_{\max} \mathrm{id}_D}$ is

$$\begin{tikzcd} C_{\phi \otimes_{\max} \mathrm{id}_D} \arrow[r] \arrow[d] & C_0(0,1] \otimes B \otimes_{\max} D \arrow[d, "\mathrm{ev}_1 \otimes \mathrm{id}_B \otimes_{\max} \mathrm{id}_D"] \\ A \otimes_{\max} D \arrow[r, "\phi \otimes_{\max} \mathrm{id}_D"] & B \otimes_{\max} D \end{tikzcd}$$

and Corollary 3.9 states that this pullback is $C_\phi \otimes_{\max} D$. So we must have an isomorphism $C_\phi \otimes_{\max} D \cong C_{\phi \otimes_{\max} \mathrm{id}_D}$.

For the minimal case we first have to argue that the map $\phi \otimes_{\min} \mathrm{id}_D$ exists. We know the algebraic tensor product map $\phi \odot \mathrm{id}_D : A \odot D \to B \odot D$ defined by $\phi \odot \mathrm{id}_D(a \otimes d) = \phi(a) \otimes d$ is a $*$-homomorphism between algebras.
Let $\sigma : B \to \mathbb{B}(\mathcal{H})$ and $\pi : D \to \mathbb{B}(\mathcal{K})$ be faithful representations on two Hilbert spaces $\mathcal{H}, \mathcal{K}$. Then we have

$$\|\phi \odot \mathrm{id}_D(x)\|_{\min} = \|(\sigma \circ \phi) \odot \pi(x)\|_{\mathbb{B}(\mathcal{H} \otimes \mathcal{K})} \quad \text{for all } x \in A \odot D$$

Observe that $\sigma \circ \phi : A \to \mathbb{B}(\mathcal{H})$ is a representation, not necessarily faithful. The minimal norm has an alternative definition using the supremum over all (not necessarily faithful) representations.

$$||x||_{\min} = \sup\left\{ \|\sigma \odot \pi(x)\|_{\mathbb{B}(\mathcal{H}' \otimes \mathcal{K}')} \mid \sigma : A \to \mathbb{B}(\mathcal{H}'), \pi : D \to \mathbb{B}(\mathcal{K}') \right\}$$

Thus we must have $||\phi \odot \mathrm{id}_D(x)||_{\min} \leq ||x||_{\min}$ for all $x \in A \odot D$. This ensures that $\phi \odot \mathrm{id}_D$ is bounded and uniquely extends to $\phi \otimes_{\min} \mathrm{id}_D : A \otimes_{\min} D \to B \otimes_{\min} D$. This same argument can be used to prove that every tensor product map $\varphi \odot \psi : A \odot B \to C \odot D$ extends uniquely to $\phi \otimes_{\min} \psi : A \otimes_{\min} B \to C \otimes_{\min} D$. This property is analogues to the universal property for the maximal tensor product, albeit a weaker property since it concerns tensor product maps and not general $*$-homomorphisms.
Lets now prove that $C_\phi \otimes_{\min} D \cong C_{\phi \otimes_{\min} \mathrm{id}_D}$. We have the following commutative diagram with exact rows from Proposition 3.8

$$\begin{tikzcd} 0 \arrow[r] & SB \arrow[r, "j"] \arrow[d, equal] & C_\phi \arrow[r, "\alpha"'] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\phi"] \arrow[l, bend right, "\pi"'] & 0 \\ 0 \arrow[r] & SB \arrow[r, "i"] & CB \arrow[r, "\eta"] & B \arrow[r] & 0 \end{tikzcd}$$

The bottom row is the "suspension-cone" exact sequence and $j$ is the inclusion $f \mapsto (0, f)$. We define the map $\pi : A \to C_\phi$ by $\pi(a) = (a, f_{\phi(a)})$ with $f(t) = (1-t)\phi(a)$. To achieve our goal, we can again use Proposition 3.8 with the diagram above in mind. We now want to define maps in the following diagram that makes it commutative and exact.

$$\begin{tikzcd}
0 \arrow[r] & S(B \otimes_{\min} D) \arrow[r, "j'"] \arrow[d, equal] & C_\phi \otimes_{\min} D \arrow[r, "\alpha'"] \arrow[d, "\sigma'"] & A \otimes_{\min} D \arrow[r] \arrow[d, "\phi \otimes_{\min} \mathrm{id}_D"] \arrow[l, bend right, "\pi'"'] & 0 \\
0 \arrow[r] & S(B \otimes_{\min} D) \arrow[r, "i'"] & C(B \otimes_{\min} D) \arrow[r, "\eta'"] & B \otimes_{\min} D \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact since this suspension-cone sequence is exact for all $C^*$-algebras. To refresh our memory, we have $C_\phi = \{(a, f) \in A \oplus CB \mid \phi(a) = f(0)\}$. The isomorphism $S(B \otimes_{\min} D) \cong C_0((0,1), B \otimes_{\min} D) \cong C_0(0,1) \otimes_{\min} B \otimes_{\min} D \cong SB \otimes_{\min} D$ gives rise to the map $j' : f \otimes d \mapsto (0, f) \otimes d$. Define $\alpha'$ by $(a, f) \otimes d \mapsto a \otimes d$, $\sigma'$ by $(a, f) \otimes d \mapsto f(t) \otimes d$ and $\pi'$ by $a \otimes d \mapsto \pi(a) \otimes d = (a, f_{\phi(a)}) \otimes d$. It is clear that the left and right quadrants commute and that $j'$, $\alpha'$ are injective and surjective respectively. We now want to show exactness of the top row. It is a fact that

$$\frac{C_\phi \otimes_{\min} D}{j'(S(B \otimes_{\min} D))} \cong A \otimes_\beta D$$

for some $C^*$-norm $\beta$ on $A \odot D$. If we can prove that $||\cdot||_{\min} = ||\cdot||_\beta$, then we have exactness of the top row [BrownOzawa, Proposition 3.7.2]. Let $y = \sum_{j=1}^n a_j \otimes d_j \in A \odot D$. We want to prove that $||y||_\beta \leq ||y||_{\min}$ since $||y||_\beta \geq ||y||_{\min}$ is given by the Takesaki Theorem [BrownOzawa, Theorem 3.4.8]. The quotient norm is given by

$$\begin{aligned}
||y||_\beta = \inf_{f \in S(B \otimes_{\min} D)} \{||\pi'(y) + j'(f)||_{\min}\} &\leq ||\pi'(y)||_{\min} \\
&= ||\sum_{i=1}^n (a_i, f_{\phi(a_i)}) \otimes d_i||_{C_\phi \otimes_{\min} D}
\end{aligned}$$

We have $C_\phi \otimes_{\min} D \subseteq (A \oplus CB) \otimes_{\min} D = (A \otimes_{\min} D) \oplus (CB \otimes_{\min} D)$ and this implies that the above is equal to

$$\begin{aligned}
&\left\| \sum_{i=1}^n (a_i \otimes d_i, f_{\phi(a_i)} \otimes d_i) \right\| |_{(A \otimes_{\min} D) \oplus (CB \otimes_{\min} D)} \\
&= \left\| \left( \sum_{i=1}^n a_i \otimes d_i, \sum_{i=1}^n f_{\phi(a_i)} \otimes d_i \right) \right\|_{(A \otimes_{\min} D) \oplus (CB \otimes_{\min} D)} \\
&= \max \left\{ \left\| \sum_{i=1}^n a_j \otimes d_j \right\|_{A \otimes_{\max} D}, \left\| \sum_{i=1}^n f_{\phi(a_j)} \otimes d_j \right\|_{CB \otimes_{\min} D} \right\}
\end{aligned}$$

Observe that $||y||_{\min}$ is in the left argument of the max expression, if we prove that the right argument is smaller that the left argument, then we are done.

Since we have an isomorphism $CB \otimes_{\min} D \cong C(B \otimes_{\min} D)$, then we have the following estimate.

$$\begin{aligned}
\left\| \sum_{i=1}^{n} f_{\phi(a_i)} \otimes d_i \right\|_{CB\otimes_{\min} D} &= \sup_{t\in[0,1)} \left\| \sum_{i=1}^{n} (1-t)\phi(a_i) \otimes d_i \right\|_{B\otimes_{\min} D} \\
= \sup_{t\in[0,1)} |1-t| \ \left\| \sum_{i=1}^{n} \phi(a_i) \otimes d_i \right\|_{B\otimes_{\min} D} &= \left\| \sum_{i=1}^{n} \phi(a_i) \otimes d_i \right\|_{B\otimes_{\min} D} \\
\leq \left\| \sum_{i=1}^{n} a_i \otimes d_i \right\|_{A\otimes_{\min} D} &= ||y||_{\min}
\end{aligned}$$

Where the last inequality comes from the fact that $\phi \otimes_{\min} \mathrm{id}_D$ is contractible. We conclude that the top row is exact and Proposition 3.8 states that $C_\phi \otimes_{\min} D \cong C_{\phi\otimes_{\min}\mathrm{id}_D}$. □

## 4 The Künneth Theorem for $\otimes_{\min}$ and $\otimes_{\max}$

In this chapter, we define the Künneth map and establish a characterization for a $C^*$-algebra $A$ belonging to the class of $C^*$-algebras that satisfy the Künneth short exact sequence for all $B$ (in both the minimal and maximal tensor product).
As we saw in the original statement of [Schochet] in the introduction, we need a new notation and terminology to simplify our expressions. This notation will primarily be used describe the big picture connections, the proofs will not have an emphasis on condensed notation in this way.

**Definition 4.1** ($\mathbb{Z}/2$-grading notation)**.** Define the notation $K_*(A) := K_0(A) \oplus K_1(A)$ for all $C^*$-algebras $A$. We associate a tensor product involving $K_0$ and $K_1$ groups with a grading in $\mathbb{Z}/2$ which is the sum of all its $K$-theory indexes modulus 2. This means that each summand of

$$\begin{aligned}
K_*(A) \otimes K_*(B) &\cong (K_0(A) \oplus K_1(A)) \otimes (K_0(B) \oplus K_1(B)) \\
= (K_0(A) \otimes K_0(B)) \oplus (K_0(A) \otimes K_1(B)) &\oplus (K_1(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B))
\end{aligned}$$

has corresponding degree $0, 1, 1, 0$ in that order. A homomorphism from a direct sum of tensor products of $K_0$, $K_1$ to a direct sum of tensor products of $K_0$, $K_1$ is said to be even if it preserves the grading and odd if it reverses the grading. So an even homomorphism sends degrees 0 to 0 and 1 to 1 and an odd homomorphism sends degrees 0 to 1 and 1 to 0. We use the same notation and terminology with $\mathrm{Tor}(K_*(A), K_*(B))$ since it also commutes with direct sums.

If $\varphi : A \to B$ is a $*$-homomorphism between $C^*$-algebras, then it induces the homomorphism $\varphi_* := K_0(\varphi) \oplus K_1(\varphi)$. $\varphi_*$ is even unless $B$ introduces a degree shift, which is something that can happen, and one such example is in the proof of this important lemma.

**Lemma 4.2.** *Let $B$ be a any $C^*$-algebra. Then there is a commutative $F = C_0(Y)$ where $Y$ is a disjoint union of lines and planes, and a $*$-homomorphism $\varphi : F \to S^2B \otimes \mathcal{K}$ such that $\varphi_* : K_*(F) \to K_*(S^2B) \cong K_*(B)$ is surjective. If $K_*(B)$ is finitely generated, then $K_*(F)$ may be required to be finitely generated too. If $K_*(B)$ is free then $\varphi_*$ is an isomorphism.*

*Proof.* [Blackadder, Proposition 23.5.1] and [Schochet, Lemma 3.1]. First choose a set of generators for $K_1(SB) \cong K_0(B)$. Let $\lambda \in K_1(SB)$ be a generator, then we know that there is a unitary $u \in M_n(SB)^+$ for some $n$ with $\pi(u) = 1$ (scalar map) and with $[u]_1 = \lambda$. Now we want to define a $*$-homomorphism $C_0(\mathbb{R}) \to M_n(SB)$ by $f \mapsto f(u)$ which in $K$-theory will hit the generator. Observe that $C_0(S^1 \setminus \{1\})$ is a $C^*$-subalgebra of $C(S^1)$ which is isomorphic to $C_0(\{z-1 \mid z \in S^1\} \setminus \{0\})$, the isomorphism sends $f \in C_0(S^1 \setminus \{1\})$ to $\tilde{f} \in C_0(\{z-1 \mid z \in S^1\} \setminus \{0\})$ given by $\tilde{f}(t) = f(t+1)$ for $t \in \{z-1 \mid z \in S^1\} \setminus \{0\}$. In the other direction we have $f(t) = \tilde{f}(t-1)$ for $t \in S^1 \setminus \{1\}$. $u-1$ is a normal element in $M_n(SB)$ and $\tilde{f}(u-1) \in M_n(SB)$ for all $f \in C_0(S^1 \setminus \{1\})$. This means that $f(u) = \tilde{f}(u-1) \in M_n(SB)$. So we have a $*$-homomorphism $\varphi' : C_0(S^1 \setminus \{1\}) \to M_n(SB)$. Since $C_0(S^1 \setminus \{1\})^+ = C(S^1)$ we have that $K_1(C_0(S^1 \setminus \{1\})) = K_1(C(S^1))$ and $(z \mapsto z) \in C(S^1)$ is the canonical generator, so $K_1(\varphi')([z \mapsto z]_1) = [u]_1 = \lambda$ (as a note, we have $C_0(S^1 \setminus \{1\}) \cong C_0(\mathbb{R})$). If we do this process for each generator, then we are able to write $F_0 = C_0(X_0)$ for some space $X_0$ which is a direct sum of lines, one for each generator. This gives us a $*$-homomorphism $\varphi_0 : F_0 \to SB \otimes \mathcal{K}$ with the induced surjection $K_1(\varphi_0) : K_1(F_0) \to K_1(SB \otimes \mathcal{K}) \cong K_1(SB) \cong K_0(B)$.

To get the map for $K_1(B)$, we can do the same procedure, but this time on $S^2B$ since $K_1(B) \cong K_1(S^2B)$. We get $F_1 = C_0(X_1)$ where $X_1$ is a disjoint union of lines with $*$-homomorphism $\varphi_1 : F_1 \to S^2B \otimes \mathcal{K}$ and surjection $K_1(\varphi_1) : K_1(F_1) \to K_1(S^2B \otimes \mathcal{K}) \cong K_1(B)$. Now we have

$$S\varphi_0 \oplus \varphi_1 : SF_0 \oplus F_1 \to S^2B \otimes \mathcal{K} \oplus S^2B \otimes \mathcal{K}$$

and since $(S^2B \otimes \mathcal{K}) \oplus (S^2B \otimes \mathcal{K}) \subseteq M_2(S^2B \otimes \mathcal{K}) \cong S^2B \otimes \mathcal{K}$. Then we have the $*$-homomorphism

$$\varphi : SF_0 \oplus F_1 \to S^2B \otimes \mathcal{K}$$

and additionally we have $K_1(SF_0) = 0 = K_0(F_1)$ since $K_1(C_0(\mathbb{R}^2)) = K_1(\mathbb{C}) =$

$K_0(C_0(\mathbb{R}))$ and $K_1(\mathbb{C}) = 0$. Finally, this gives us the desired mapping

$$\begin{aligned}\varphi_* : K_0(SF_0 \oplus F_1) \oplus K_1(SF_0 \oplus F_1) &\cong K_0(SF_0) \oplus K_1(F_1)\\ \cong K_1(F_0) \oplus K_1(F_1) &\xrightarrow{K_1(\varphi_0) \oplus K_1(\varphi_1)} K_1(SB \otimes \mathcal{K}) \oplus K_1(S^2B \otimes \mathcal{K})\\ &\cong K_0(B \otimes \mathcal{K}) \oplus K_1(B \otimes \mathcal{K})\end{aligned}$$

and it is surjective. □

We will now define the Künneth map [Schochet, Equation 2.1] $\alpha_\gamma : K_*(A) \otimes K_*(B) \to K_*(A \otimes_\gamma B)$ for $\gamma = \min/\max$. We start by defining the map $\alpha' : K_0(A) \otimes K_0(B) \to K_0(A \otimes_\gamma B)$, then later we can expand to the whole $\alpha_\gamma$. Let us start off in the simple setting where $A, B$ are unital. The idea is define a homomorphism where the projections are mapped in a canonical way

$$[p]_0 \otimes [q]_0 \mapsto [p \otimes q]_0 \text{ for all } p \in \mathcal{P}(M_n(A)),\ q \in \mathcal{P}(M_m(B))$$

since there is an isomorphism $M_n(A) \otimes_\gamma M_m(B) \cong M_{nm}(A \otimes_\gamma B)$. To sketch the construction of this map, start with $\varphi : \mathcal{D}(A) \times \mathcal{D}(B) \to K_0(A \otimes_\gamma B)$ defined by $([p], [q]) \mapsto [p \otimes q]_0$ for $p \in \mathcal{P}(M_n(A))$, $q \in \mathcal{P}(M_m(B))$. This map is bilinear (bi-additive in this case) since for $p, p' \in \mathcal{P}(M_n(A))$ and $q \in \mathcal{P}(M_m(B))$, we have

$$\begin{aligned}([p] + [p'], [q]) = ([p \oplus p'], [q]) &\mapsto [(p \oplus p') \otimes q]_0\\ &= [(p \otimes q) \oplus (p' \otimes q)]_0 = [p \otimes q]_0 + [p' \otimes q]_0\end{aligned}$$

and similarly for the second argument. It is also well-defined in that it respects equivalence of projections. Indeed, if $p \sim_0 p'$ for $p, p' \in \mathcal{P}(M_n(A))$ and $q \sim_0 q'$ for $q, q' \in \mathcal{P}(M_m(B))$, then we have $v^*v = p$, $vv^* = p'$ for some $v \in M_n(A)$ and $w^*w = q$, $ww^* = q'$ for some $w \in M_m(B)$. Finally we have $p \otimes q \sim_0 p' \otimes q'$ since $(v \otimes w)^*(v \otimes w) = p \otimes q$ and $(v \otimes w)(v \otimes w)^* = p' \otimes q'$ in $\mathcal{P}(M_{nm}(A \otimes_\gamma B))$.
We are now going to use the Grothendieck construction (and universal property) on $\mathcal{D}(A)$ and $\mathcal{D}(B)$ ($\mathcal{D}$(A) is the abelian semigroup $\mathcal{P}_\infty(A)/\sim_0$ where $[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}$). We do this because the above map is bilinear, thus fixing one argument induces an additive map. Let $[q] \in \mathcal{D}(B)$, then $\mathcal{D}(A) \to K_0(A \otimes_\gamma B)$ produced by $[p] \mapsto [p \otimes q]_0$ is additive. From the universal construction, we have a unique map $K_0(A) \to K_0(A \otimes_\gamma B)$ given by $[p]_0 - [p']_0 \mapsto [p \otimes q]_0 - [p' \otimes q]_0$. This gives us a bilinear map $K_0(A) \otimes \mathcal{D}(B) \to K_0(A \otimes_\gamma B)$ given by $([p]_0 - [p]_0, [q]) \mapsto [p \otimes q]_0 - [p' \otimes q]_0$. Again, fix $[p]_0 - [q]_0 \in K_0(A)$, now we have an additive map $\mathcal{D}(B) \to K_0(A \otimes_\gamma B)$. Using the Grothendieck construction again we get a bilinear map $K_0(A) \times K_0(B) \to$

$K_0(A \otimes_\gamma B)$ with

$$\begin{aligned}([p]_0 - [p']_0, [q]_0 - [q']_0) &= ([p]_0, [q]_0 - [q']_0) - ([p']_0, [q]_0 - [q']_0) \\ &= ([p]_0, [q]_0) - ([p]_0, [q']_0) - ([p]_0, [q]_0) + ([p']_0, [q']_0) \\ &\mapsto [p \otimes q]_0 - [p \otimes q']_0 - [p' \otimes q]_0 + [p' \otimes q']_0\end{aligned}$$

Finally, the universal property of tensor products gives us the homomorphism $\alpha' : K_0(A) \otimes K_0(B) \to K_0(A \otimes_\gamma B)$ with the desired property. The map is easily seen to be natural.
In the non-unital case we will need the following lemma.

**Lemma 4.3.** *Let $A, B$ be two $C^*$-algebras and the tensor norm $\gamma = \min/\max$. If $x \in K_0(A^+ \otimes_\gamma B^+)$, then $x \in K_0(A \otimes_\gamma B)$ if and only if*

$$\psi := (\mathrm{id}_{A^+} \otimes \pi_{B^+}) \oplus (\pi_{A^+} \otimes \mathrm{id}_{B^+}) : A^+ \otimes_\gamma B^+ \to (A^+ \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes B^+)$$

*has $\psi(x) = 0$.*

*Proof.* Let $x \in K_0(A^+ \otimes_\gamma B^+)$, then we have $p, q \in \mathcal{P}(M_n(A^+ \otimes_\gamma B^+))$ with $x = [p]_0 - [q]_0$. If $K_0(\psi)(x) = ([\mathrm{id}_{A^+} \otimes \pi_{B^+}(p)]_0 - [\mathrm{id}_{A^+} \otimes \pi_{B^+}(q)]_0, [\pi_{A^+} \otimes \mathrm{id}_{B^+}(p)]_0 - [\pi_{A^+} \otimes \mathrm{id}_{B^+}(q)]_0) = 0$, then the first coordinate has $[\mathrm{id}_{A^+} \otimes \pi_{B^+}(p)]_0 - [\mathrm{id}_{A^+} \otimes \pi_{B^+}(q)]_0 = 0$. Observe that the sequence

$$0 \longrightarrow A^+ \otimes_\gamma B \longrightarrow A^+ \otimes_\gamma B^+ \xrightarrow{\mathrm{id}_{A^+} \otimes \pi_{B^+}} A^+ \otimes \mathbb{C} \longrightarrow 0$$

splits and so the K-theory splits, this implies that $x \in K_0(A^+ \otimes_\gamma B)$. In addition, we have $[\pi_{A^+} \otimes \mathrm{id}_{B^+}(p)]_0 - [\pi_{A^+} \otimes \mathrm{id}_{B^+}(q)]_0 = 0$ and from the split exact sequence

$$0 \longrightarrow A \otimes_\gamma B \longrightarrow A^+ \otimes_\gamma B \xrightarrow{\pi_{A^+} \otimes \mathrm{id}_B} \mathbb{C} \otimes B \longrightarrow 0$$

we again see that this implies $x \in K_0(A \otimes_\gamma B)$.
If $x \in K_0(A \otimes B) \subseteq K_0(A^+ \otimes B^+)$, then $x \in K_0(A^+ \otimes B)$ and so $K_0(\pi \otimes \mathrm{id}_B)(x) = 0$. Again if $x \in K_0(A^+ \otimes B^+)$, then from the previous inclusion we have $K_0(\mathrm{id}_{A^+} \otimes \pi)(x) = 0$. ☐

Now we are equipped to handle the non-unital case for the Künneth map. Let $A, B$ be $C^*$-algebras and let $x \in K_0(A)$ and $y \in K_0(B)$. From the standard picture of $K_0$ [Rørdam, Proposition 4.2.2], we have $p_1, p_2 \in \mathcal{P}(M_n(A^+))$ with $p_1 - p_2 \in M_n(A)$ and $q_1, q_2 \in \mathcal{P}(M_m(B^+))$ with $q_1 - q_2 \in M_m(B)$ such that $x = [p_1]_0 - [p_2]_0$ and $y = [q_1]_0 - [q_2]_0$. We can now use the Künneth map to send $x \otimes y$ into $K_0(A^+ \otimes_\gamma B^+)$.

As we saw before we have

$$([p_1]_0 - [p_2]_0) \otimes ([q_1]_0 - [q_2]_0) \mapsto$$
$$[p_1 \otimes q_1]_0 - [p_1 \otimes q_2]_0 - [p_2 \otimes q_1]_0 + [p_2 \otimes q_2]_0 := z \in K_0(A^+ \otimes_\gamma B^+)$$

It turns out that this element actually lands in $K_0(A \otimes_\gamma B)$. See this by using $\psi$ from Lemma 4.3

$$K_0(\psi)(z) = ([p_1 \otimes \pi(q_1)]_0 - [p_1 \otimes \pi(q_2)]_0 - [p_2 \otimes \pi(q_1)]_0 + [p_2 \otimes \pi(q_2)]_0,$$
$$[\pi(p_1) \otimes q_1]_0 - [\pi(p_1) \otimes q_2]_0 - [\pi(p_2) \otimes q_1]_0 + [\pi(p_2) \otimes q_2]_0) = 0$$

since $\pi(p_1) = \pi(p_2)$ and $\pi(q_1) = \pi(q_2)$. The lemma states that $x \in K_0(A \otimes_\gamma B)$ and thus we get a well defined map even in the non-unital case.
The standard $K_0$ picture of the Künneth map for $C^*$-algebras $A$ and $B$ maps $x \otimes y \in K_0(A) \otimes K_0(B)$, which can be written $x = [p_1]_0 - [p_2]_0$, $y = [q_1]_0 - [q_2]_0$ for $p_1, p_2 \in \mathcal{P}_n(A^+)$ and $q_1, q_2 \in \mathcal{P}_m(B^+)$ with $p_1 - p_2 \in M_n(A)$ and $q_1 - q_2 \in M_m(B)$, to the element

$$[p_1 \otimes q_1]_0 - [p_1 \otimes q_2]_0 - [p_2 \otimes q_1]_0 + [p_2 \otimes q_2]_0 \in K_0(A \otimes_\gamma B)$$

This can now be extended to get the full Künneth homomorphism for all $K_i$ groups.

**Definition 4.4** (The Künneth map)**.** Let $A, B$ be $C^*$-algebras. Then the Künneth homomorphism $\alpha_\gamma^{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes_\gamma B)$, where $\gamma = \min/\max$, is defined in the $K_0$ case and extended in the following way

$$K_0(A) \otimes K_0(B) \xrightarrow{\alpha'} K_0(A \otimes_\gamma B)$$
$$K_1(A) \otimes K_1(B) \cong K_0(SA) \otimes K_0(SB) \xrightarrow{\alpha'} K_0(SA \otimes_\gamma SB) \cong K_0(A \otimes_\gamma B)$$
$$K_1(A) \otimes K_0(B) \cong K_0(SA) \otimes K_0(B) \xrightarrow{\alpha'} K_0(SA \otimes_\gamma B) \cong K_1(A \otimes_\gamma B)$$
$$K_0(A) \otimes K_1(B) \cong K_0(A) \otimes K_0(SB) \xrightarrow{\alpha'} K_0(A \otimes_\gamma SB) \cong K_1(A \otimes_\gamma B)$$

The Künneth map is natural because $\alpha'$ and the isomorphisms $K_1(*) \cong K_0(S*)$ and $K_1(S*) \cong K_0(*)$ are natural.

Let $A$ be a $C^*$-algebra. The following result relates the existence of the Künneth short exact sequence for all $C^*$-algebras $B$ to two equivalent conditions, one of which does not involve the Künneth map explicitly.

**Theorem 4.5** (The Künneth Theorem for $\otimes_{\min}$ and $\otimes_{\max}$)**.** *Let $A$ be a $C^*$-algebra and let $\gamma = \min/\max$ be the chosen $C^*$-norm on the tensor product. Then the following conditions are equivalent:*

1. *For any $C^*$-algebra $B$, if $K_*(B) = 0$ then $K_*(A \otimes_\gamma B) = 0$*

2. *For any $C^*$-algebra $B$, if $K_*(B)$ is free then the even Künneth map*

$$\alpha_\gamma^{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes_\gamma B)$$

*is an isomorphism.*

3. *For any $C^*$-algebra $B$, there is a natural short exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\alpha_\gamma^{A,B}} K_*(A \otimes_\gamma B) \xrightarrow{\beta_\gamma^{A,B}} \mathrm{Tor}(K_*(A), K_*(B)) \to 0$$

*where the Künneth map $\alpha_\gamma^{A,B}$ is even and $\beta_\gamma^{A,B}$ is odd.*

*Proof.* For the implication *1.* $\Rightarrow$ *2.* see [Uuye, Theorem 3.1] and for *2.* $\Rightarrow$ *3.* see [Schochet]. This proof is almost exactly the same for both the minimal and maximal case, we will therefore use $\otimes$ instead of $\otimes_{\min}$ or $\otimes_{\max}$. If two separate arguments are needed for the two cases, then they will be given in a clearly delineated way before we push on using $\otimes$. The implications *3.* $\Rightarrow$ *2.* $\Rightarrow$ *1.* are trivial. To prove equivalence between the statements, we have to prove that *1.* $\Rightarrow$ *2.* and *2.* $\Rightarrow$ *3.*

*1.* $\Rightarrow$ *2.*
Let $B$ be a $C^*$-algebra with free $K_*(B)$, that is $K_*(B)$ is a free abelian group. Lemma 4.2 states that we can find a $*$-homomorphism $\varphi : D \to S^2B \otimes \mathcal{K}$ with isomorphism $\varphi_* : K_*(D) \to K_*(S^2B \otimes \mathcal{K})$. Let $C_\varphi$ be the mapping cone for $\varphi$, then by Corollary 3.7 we have that $K_*(C_\varphi) = 0$. Since $A \otimes C_\varphi \cong C_{\mathrm{id}_A \otimes \varphi}$ from Proposition 3.10, then we can write

$$K_*(C_{\mathrm{id}_A \otimes \varphi}) \cong K_*(A \otimes C_\varphi) = 0$$

using condition 1. Thus Corollary 3.7 states that $\mathrm{id}_A \otimes \varphi$ is an isomorphism in K-theory. Observe the following commutative diagram.

$$\begin{tikzcd}
K_*(A) \otimes K_*(D) \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{K_*(A)} \otimes \varphi_*"] & K_*(A \otimes D) \arrow[d, "(\mathrm{id}_A \otimes \varphi)_*"] \\
K_*(A) \otimes K_*(S^2B \otimes \mathcal{K}) \arrow[r, "\alpha"] & K_*(A \otimes S^2B \otimes \mathcal{K})
\end{tikzcd}$$

The sides are both isomorphisms of groups. If the top map is an isomorphism, then we know that $\alpha$ is as well. We can write $D = (\oplus_{\Lambda_1} C_0(\mathbb{R})) \oplus (\oplus_{\Lambda_2} C_0(\mathbb{R}^2))$ for some index sets $\Lambda_1, \Lambda_2$ and this is key in showing that

$$\begin{aligned}
K_0(A \otimes D) &\cong K_0(A \otimes (\oplus_{\Lambda_1} C_0(\mathbb{R}))) \oplus K_0(A \otimes (\oplus_{\Lambda_2} C_0(\mathbb{R}^2))) \\
&\cong (\oplus_{\Lambda_2} K_0(A \otimes C_0(\mathbb{R}))) \oplus (\oplus_{\Lambda_2} K_0(A \otimes C_0(\mathbb{R}^2))) \\
&\cong (\oplus_{\Lambda_1} K_1(A)) \oplus (\oplus_{\Lambda_2} K_0(A)) \\
&\cong (\oplus_{\Lambda_1} (K_1(A) \otimes \mathbb{Z})) \oplus (\oplus_{\Lambda_2} (K_0(A) \otimes \mathbb{Z})) \\
&\cong (K_1(A) \otimes K_1(D)) \oplus (K_0(A) \otimes K_0(D))
\end{aligned}$$

The same canonical isomorphisms can be used to prove $K_1(A \otimes D) \cong (K_0(A) \otimes K_1(D)) \oplus (K_1(A) \otimes K_0(D))$. This gives us the isomorphism $K_*(A) \otimes K_*(D) \cong K_*(A \otimes D)$. Now we know that $\alpha$ is an isomorphism and this gives us an isomorphism $K_*(A) \otimes K_*(B) \cong K_*(A \otimes B)$ since Bott periodicity shifts gives $K_*(S^2B \otimes \mathcal{K}) \cong K_*(B)$ and $K_*(A \otimes S^2B \otimes \mathcal{K}) \cong K_*(A \otimes B)$.

*2. ⇒ 3.*

To start with, let $B$ be a $C^*$-algebra satisfying statement *2*. Lemma 4.2 gives us a $*$-homomorphism $\mu : F \to S^2B \otimes \mathcal{K}$ such that $\mu_* : K_*(F) \to K_*(B \otimes \mathcal{K})$ is surjective. Let $C_\mu$ be the mapping cone of $\mu$ and let us investigate the induced mapping cone sequence from the remark below Definition 3.6.

$$0 \longrightarrow S(S^2B \otimes \mathcal{K}) \longrightarrow C_\mu \xrightarrow{\nu} F \longrightarrow 0$$

$C_\mu$ is a pullback and this means that Proposition 3.8 gives us the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S(S^2B \otimes \mathcal{K}) \arrow[r] \arrow[d, equal] & C_\mu \arrow[r, "\nu"] \arrow[d] & F \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & S(S^2B \otimes \mathcal{K}) \arrow[r] & C(S^2B \otimes \mathcal{K}) \arrow[r] & S^2B \otimes \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Observe that $C(S^2B \otimes \mathcal{K})$ is contractible, i.e it is homotopic to a point. Since $K$-theory is homotopy invariant, we get that $K_*(C(S^2B \otimes \mathcal{K})) = 0$. The boundary maps are natural, and so for $q \in \mathbb{Z}/2$, we have the following commutative diagram with exact rows. (with some unnecessary suspensions removed with Bott periodicity)

$$\begin{tikzcd}
K_q(SB \otimes \mathcal{K}) \arrow[r] \arrow[d, equal] & K_q(C_\mu) \arrow[r, "\nu_*"] \arrow[d] & K_q(F) \arrow[r, "\delta_1"] \arrow[d, "\mu_*"] & K_{1-q}(SB \otimes \mathcal{K}) \arrow[d, equal] \\
K_q(SB \otimes \mathcal{K}) \arrow[r] & K_q(CB \otimes \mathcal{K}) = 0 \arrow[r] & K_q(B \otimes \mathcal{K}) \arrow[r, "\delta_2"] & K_{1-q}(SB \otimes \mathcal{K})
\end{tikzcd}$$

This diagram can be extended to the right and the left in the obvious way. Exactness forces $\delta_2$ to be an isomorphism, since $K_{q-1}(CB \otimes \mathcal{K}) = 0$ as well in the extended diagram, which in turn makes $\delta_1$ surjective since $\mu_*$ is surjective. The homomorphism $K_q(SB \otimes \mathcal{K}) \to K_q(C_\mu)$ is the zero map since we can extend the diagram to the left by a surjective boundary map to $K_q(SB \otimes \mathcal{K})$. To summarize, we have

$$\begin{tikzcd}
K_q(SB \otimes \mathcal{K}) \arrow[r, "0"] \arrow[d, equal] & K_q(C_\mu) \arrow[r, tail, "\nu_*"] \arrow[d] & K_q(F) \arrow[r, two heads, "\delta_1"] \arrow[d, "\mu_*"] & K_{1-q}(SB \otimes \mathcal{K}) \arrow[d, equal] \\
K_q(SB \otimes \mathcal{K}) \arrow[r] & K_q(CB \otimes \mathcal{K}) = 0 \arrow[r] & K_q(B \otimes \mathcal{K}) \arrow[r, "\delta_2(\cong)"] & K_{1-q}(SB \otimes \mathcal{K})
\end{tikzcd}$$

and so for $q \in \mathbb{Z}/2$, then we have the free resolution of $K_q(B \otimes \mathcal{K})$

$$0 \longrightarrow K_q(C_\mu) \xrightarrow{\nu_*} K_q(F) \xrightarrow{\mu_*} K_q(B \otimes \mathcal{K}) \longrightarrow 0$$

We now settle this part of the proof and continue with the construction of our desired short exact sequence. The short exact sequence $0 \to S(S^2B \otimes \mathcal{K}) \to C_\mu \xrightarrow{\nu} F \to 0$ can be tensorized with $A$ to produce the short exact sequence

$$0 \longrightarrow A \otimes_\alpha (S(S^2B \otimes \mathcal{K})) \longrightarrow A \otimes_\alpha C_\mu \xrightarrow{\mathrm{id}_A \otimes \nu} A \otimes_\alpha F \longrightarrow 0$$

Observe that this works for $\alpha = \min/\max$ since $F$ is nuclear and using [BrownOzawa, Proposition 3.7.2]. The six-term exact sequence of the above gives us the following exact sequence

$$\begin{tikzcd}
K_q(A \otimes C_\mu) \arrow[r, "{(\mathrm{id}_A \otimes \nu)_*}^q"] & K_q(A \otimes F) \arrow[r] & K_q(A \otimes B) = K_{q-1}(A \otimes SB) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
K_{q-1}(A \otimes C_\mu) \arrow[r, "{(\mathrm{id}_A \otimes \nu)_*}^{q-1}"] & K_{q-1}(A \otimes F) &
\end{tikzcd} \tag{$\star$}$$

for $q \in \mathbb{Z}/2$. The Künneth short exact sequence in *3.* can be decomposed into two exact sequences, one with $K_0(A \otimes B)$ in the middle and one with $K_1(A \otimes B)$ in the middle. We will start be constructing the exact sequence for $K_0(A \otimes B)$. In this case, let $q = 0$ in the above diagram and observe that if we unsplice the sequence, we get the exact sequence

$$0 \longrightarrow \operatorname{coker}({(\mathrm{id}_A \otimes \nu)_*}^0) \longrightarrow K_0(A \otimes B) \longrightarrow \ker({(\mathrm{id}_A \otimes \nu)_*}^1) \longrightarrow 0$$

Since $K_*(C_\mu)$ and $K_*(F)$ are free, then by condition *2.* we have that

$$K_0(A \otimes C_\mu) \cong \begin{smallmatrix} K_0(A) \otimes K_0(C_\mu) \\ \oplus \\ K_1(A) \otimes K_1(C_\mu) \end{smallmatrix} \qquad K_0(A \otimes F) \cong \begin{smallmatrix} K_0(A) \otimes K_0(F) \\ \oplus \\ K_1(A) \otimes K_1(F) \end{smallmatrix}$$

via the Künneth map $\alpha$. Naturality of $\alpha$ gives us the commutative square in the diagram

$$\begin{tikzcd}
 & K_0(A \otimes C_\mu) \arrow[r, "(\mathrm{id}_A \otimes \nu)^0_*"] & K_0(A \otimes F) & \\
\substack{\mathrm{Tor}(K_0(A),K_0(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_1(B))} \arrow[r, tail, "\delta_1"] & \substack{K_0(A)\otimes K_0(C_\mu) \\ \oplus \\ K_1(A)\otimes K_1(C_\mu)} \arrow[u, "\alpha"] \arrow[r, "\substack{\mathrm{id}\otimes\nu^0_* \\ \oplus \\ \mathrm{id}\otimes\nu^1_*}"] & \substack{K_0(A)\otimes K_0(F) \\ \oplus \\ K_1(A)\otimes K_1(F)} \arrow[u, "\alpha"'] \arrow[r, two heads, "\substack{\mathrm{id}\otimes\mu^0_* \\ \oplus \\ \mathrm{id}\otimes\mu^1_*}"] & \substack{K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B)}
\end{tikzcd}$$

while Definition 2.6 and Proposition 2.5 uses the free resolution we made of $K_*(B)$ to gives us $\delta_1$ along with exactness of the row. In a similar fashion we get

$$\begin{tikzcd}
 & K_1(A \otimes C_\mu) \arrow[r, "(\mathrm{id}_A \otimes \nu)^1_*"] & K_1(A \otimes F) & \\
\substack{\mathrm{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_0(B))} \arrow[r, tail, "\delta_2"] & \substack{K_0(A)\otimes K_1(C_\mu) \\ \oplus \\ K_1(A)\otimes K_0(C_\mu)} \arrow[u, "\alpha"] \arrow[r, "\substack{\mathrm{id}\otimes\nu^1_* \\ \oplus \\ \mathrm{id}\otimes\nu^0_*}"] & \substack{K_0(A)\otimes K_1(F) \\ \oplus \\ K_1(A)\otimes K_0(F)} \arrow[u, "\alpha"'] \arrow[r, two heads, "\substack{\mathrm{id}\otimes\mu^1_* \\ \oplus \\ \mathrm{id}\otimes\mu^0_*}"] & \substack{K_0(A)\otimes K_1(B) \\ \oplus \\ K_1(A)\otimes K_0(B)}
\end{tikzcd}$$

which also has an exact row and commutes. The first diagram gives us

$$\operatorname{coker}((\mathrm{id}_A \otimes \nu)^0_*) \cong \operatorname{coker} \begin{pmatrix} \mathrm{id}\otimes\nu^0_* \\ \oplus \\ \mathrm{id}\otimes\nu^1_* \end{pmatrix} \cong \substack{K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B)}$$

and the second diagram gives us

$$\ker((\mathrm{id}_A \otimes \nu)^1_*) \cong \ker \begin{pmatrix} \mathrm{id}\otimes\nu^1_* \\ \oplus \\ \mathrm{id}\otimes\nu^0_* \end{pmatrix} \cong \substack{\mathrm{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_0(B))}$$

The unspliced sequence with these new isomorphisms are now

$$0 \longrightarrow \substack{K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B)} \longrightarrow K_0(A \otimes B) \longrightarrow \substack{\mathrm{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_0(B))} \longrightarrow 0$$

If we unsplice the $(\star)$ sequence for $q = 1$, then we can use the same argument on the two diagrams to get the exact sequence

$$0 \longrightarrow \substack{K_0(A)\otimes K_1(B) \\ \oplus \\ K_1(A)\otimes K_0(B)} \longrightarrow K_1(A \otimes B) \longrightarrow \substack{\mathrm{Tor}(K_0(A),K_0(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_1(B))} \longrightarrow 0$$

These sequences combine to give us the Künneth short exact sequence in statement *3*. □

The statement of Theorem 4.5 can be made where we require $B$ to be any separable $C^*$-algebra instead of just any $C^*$-algebra.

**Corollary 4.6** (Separable version of the Künneth Theorem)**.** *Let $A$ be a $C^*$-algebra and let $\gamma = \min/\max$ be the chosen $C^*$-norm on the tensor product. Then the following conditions are equivalent:*

*1'. For any separable $C^*$-algebra $B$, if $K_*(B) = 0$ then $K_*(A \otimes_\gamma B) = 0$*

*2'. For any separable $C^*$-algebra $B$, if $K_*(B)$ is free then the even Künneth map*

$$\alpha_\gamma^{A,B} : K_*(A) \otimes K_*(B) \to K_*(A \otimes_\gamma B)$$

*is an isomorphism.*

*3'. For any separable $C^*$-algebra $B$, there is a natural short exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\alpha_\gamma^{A,B}} K_*(A \otimes_\gamma B) \xrightarrow{\beta_\gamma^{A,B}} \mathrm{Tor}(K_*(A), K_*(B)) \to 0$$

*Proof.* The proof to show equivalence between statements *1', 2', 3'* is exactly the same as in the proof of Theorem 4.5. If we show that *3.* and *3'.* are equivalent, then the three statements with the weaker assumption on $B$ is equivalent to the three with the stronger assumption on $B$. *3.* clearly implies *3'.* and for the other direction, write any $C^*$-algebra $B$ as its direct limit of separable $C^*$-subalgebras $B \cong \lim_{B' \subseteq B} B'$ where $B'$ is separable. Then we have $A \otimes_\gamma B \cong \lim_{B' \subseteq B} A \otimes_\gamma B'$ (this always works in the minimal case and works in the maximal case because the connecting $*$-homomorphisms, the inclusions, are injective). Continuity of $K$-theory and the fact that inductive limits of abelian groups commute with the tensor product and torsion product gives us a *3'.* sequence for each separable $B'$. The start of the proof of Proposition 2.5 explains why the direct limit of this exact sequence is exact. This proves that *3'.* implies *3.* and we are done. $\square$

We can finish off this chapter by defining the classes that we allured to in the beginning.

**Definition 4.7.** For $\gamma = \min/\max$, let $\mathcal{N}_\gamma$ denote the class of $C^*$-algebras $A$ satisfying the equivalent conditions of Theorem 4.5.

It was shown in [Uuye, Example 5.1, 5.3] that $\mathcal{N}_{\min} \setminus \mathcal{N}_{\max} \neq \emptyset$ and also that $\mathcal{N}_{\max} \setminus \mathcal{N}_{\min} \neq \emptyset$ by constructing examples of each. So our classes are not the same.

**Lemma 4.8.** *Let $\gamma = \min/\max$. Then the classes $\mathcal{N}_\gamma$ has the following properties.*

1. *If $A, B \in \mathcal{N}_\gamma$, then $A \otimes_\gamma B \in \mathcal{N}_\gamma$.*
2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $C$ nuclear. If two of the $C^*$-algebras are in $\mathcal{N}_{\min}$, then so is the third.*
3. *If a short exact has two $C^*$-algebras that are in $\mathcal{N}_{\max}$, then so is the third.*

*Proof.*

*1.*

We will use Theorem 4.5 statement *1.* For all $C^*$-algebras $C$ with $K_*(C) = 0$, we have $0 = K_*(A \otimes_\gamma C) = K_*(B \otimes_\gamma C)$. This means that we must have $K_*((A \otimes_\gamma B) \otimes_\gamma C) = K_*(A \otimes_\gamma (B \otimes_\gamma C)) = 0$ implying $A \otimes_\gamma B \in \mathcal{N}_\gamma$.

*2.*

Using the equivalent statement *1.* from Theorem 4.5. Let $D$ be a $C^*$-algebra with $K_*(D) = 0$, since $C$ is nuclear, then by [BrownOzawa, Proposition 3.7.2] we have that

$$0 \longrightarrow A \otimes_{\min} D \longrightarrow B \otimes_{\min} D \longrightarrow C \otimes_{\min} D \longrightarrow 0$$

is exact. This short exact sequence has the six-term sequence

$$\begin{tikzcd} K_*(A \otimes_{\min} D) \arrow[r] & K_*(B \otimes_{\min} D) \arrow[r] & K_*(C \otimes_{\min} D) \arrow[d] \\ K_{1-*}(C \otimes_{\min} D) \arrow[u] & K_{1-*}(B \otimes_{\min} D) \arrow[l] & K_{1-*}(A \otimes_{\min} D) \arrow[l] \end{tikzcd}$$

where if any two of $\{A, B, C\}$ is in $\mathcal{N}_{\min}$, then every group in the sequence is forced to be the zero group and so the last is also in $\mathcal{N}_{\min}$.

*3.* This proof is exactly the same as in *2.* The only difference is that the maximal tensor product preserves exactness when tensorising an exact sequence with a $C^*$-algebra. This is why this statement does not assume nuclearity of the last term in the short exact sequence. □

## 5 $K$-theory with Coefficients

In this chapter we look at $K$-theory with coefficients following [Gabe, Chapter 2.3] closely. We start off with the definition of the main object of interest.

**Definition 5.1** (K-theory with Coefficients)**.** Let $A$ be a $C^*$-algebra. For $i \in \mathbb{Z}/2$ and $n \geq 2$ we define

$$K_i(A; \mathbb{Z}/n) := K_{1-i}(A \otimes \mathbb{I}_n)$$

where $\mathbb{I}_n := \{f \in C([0,1], M_n) \mid f(1) \in \mathbb{C}1_{M_n}, f(0) = 0\}$.

It will be proven in Lemma 5.2 that $\mathbb{I}_n$ is nuclear, this is why the tensor product has no specified $C^*$-norm. Observe that

$$A \otimes \mathbb{I}_n \cong \{f \in C([0,1], M_n(A)) \mid f(1) \in \operatorname{diag}_n(A), f(0) = 0\}$$

since the LHS has

$$\begin{aligned} A\otimes \mathbb{I}_n \hookrightarrow A\otimes M_n(C[0,1]) &\xrightarrow{\cong} M_n(A\otimes C[0,1]) \\ &\xrightarrow{\cong} M_n(C([0,1],A)) \xrightarrow{\cong} C([0,1],M_n(A)) \end{aligned}$$

and the image in $C([0,1],M_n(A))$ is precisely the RHS above.
It can be shown that $A\otimes\mathbb{I}_n$ is the mapping cone of $A\to M_n(A)$ where $a\mapsto \operatorname{diag}_n(a)$. To see this, we first have to define the following short exact sequence

$$0 \longrightarrow SM_n \xrightarrow{\mu_n} \mathbb{I}_n \xrightarrow{\nu_n} \mathbb{C} \longrightarrow 0$$

where $\mu_n$ is the inclusion and $\nu_n$ is the diagonal value at the non-trivial endpoint. Tensorising this sequence with $A$, we get another short exact sequence with $A\otimes\mathbb{I}_n$ in the middle. Considering the diagram in Proposition 3.8, use this sequence as the top sequence and use the short exact sequence $0\to SM_n(A)\to CM_n(A)\to M_n(A)\to 0$ as the bottom. Then the proposition implies that $A\otimes\mathbb{I}_n$ is the mapping cone of $A\to M_n(A)$ where $a\mapsto\operatorname{diag}_n(a)$.

**Lemma 5.2.** $\mathbb{I}_n$ *is nuclear*

*Proof.* The short exact sequence

$$0 \longrightarrow SM_n \xrightarrow{\mu_n} \mathbb{I}_n \xrightarrow{\nu_n} \mathbb{C} \longrightarrow 0$$

has $SM_n$ and $\mathbb{C}$ being nuclear. From the two out of three property of the class of nuclear $C^*$-algebras in Lemma 3.2, we see that $\mathbb{I}_n$ must be nuclear as well. □

The mapping cone sequence of $\mathbb{I}_n\otimes A$

$$0 \longrightarrow SM_n\otimes A \xrightarrow{\mu_n\otimes \mathrm{id}_A} \mathbb{I}_n\otimes A \xrightarrow{\nu_n\otimes\mathrm{id}_A} A \longrightarrow 0$$

induces the six-term exact sequence

$$\begin{tikzcd} K_0(A) \arrow[r, "\mu^{(n)}_{0,A}"] & K_0(A;\mathbb{Z}/n) \arrow[r, "\nu^{(n)}_{0,A}"] & K_1(A) \arrow[d, "\times n"] \\ K_0(A) \arrow[u, "\times n"] & K_1(A;\mathbb{Z}/n) \arrow[l, "\nu^{(n)}_{1,A}"] & K_1(A) \arrow[l, "\mu^{(n)}_{1,A}"] \end{tikzcd}$$

where $K_0(SM_n\otimes A)\cong K_1(A)$ and $K_1(SM_n\otimes A)\cong K_0(A)$ are identified by the suspension and stability isomorphisms. The maps in the six-term sequence are natural in $A$ by functoriality of $K$-theory. The boundary maps are precisely $\times n$

under our identifications. This can be seen from this commutative diagram

$$\begin{tikzcd} & K_{i-1}(SM_n \otimes A) \arrow[d, "\cong"] & \\ K_i(A) \arrow[ur, "\partial"] \arrow[r, "(\mathrm{diag}_n)_*^i"'] & K_i(M_n(A)) \arrow[r, "\cong"] & K_i(A) \end{tikzcd}$$

where the bottom composition is $\times n$. The triangle commutes from the fact that the boundary maps in the six-term sequence of a mapping cone is the same as the induced homomorphism of the map defining the mapping cone (up to isomorphism). An argument for why this is true was made the proof of *2.* $\Rightarrow$ *3.* in Theorem 4.5. The composition giving the identification $K_{1-i}(SM_n \otimes A) \cong K_i(A)$ is the "L" shaped part of the diagram.

**Definition 5.3** (Bockstein Operations)**.** Let $i \in \mathbb{Z}/2$, $n, m \geq 2$ and let $A$ be a $C^*$-algebra, then the homomorphisms

- $\mu_{i,A}^{(n)} : K_i(A) \to K_i(A; \mathbb{Z}/n)$
- $\nu_{i,A}^{(n)} : K_i(A; \mathbb{Z}/n) \to K_{1-i}(A)$
- $\kappa_{i,A}^{(nm,n)} : K_i(A; \mathbb{Z}/n) \to K_i(A; \mathbb{Z}/nm)$
- $\kappa_{i,A}^{(n,nm)} : K_i(A; \mathbb{Z}/nm) \to K_i(A; \mathbb{Z}/n)$

are called Bockstein operations. $\mu_{i,A}^{(n)}$ and $\nu_{i,A}^{(n)}$ are induced by $\mu_n$ and $\nu_n$ as described above. $\kappa_{i,A}^{(nm,n)}$ is induced by the inclusion $\kappa_{nm,n} : \mathbb{I}_n \to \mathbb{I}_{nm}$ given by $f \mapsto \mathrm{diag_m}(f)$ and $\kappa_{i,A}^{(n,nm)}$ is induced by $\kappa_{n,nm} : \mathbb{I}_{nm} \to \mathbb{I}_n \otimes M_m \cong M_m(\mathbb{I}_n)$ given by $f \mapsto f$ identification in $M_m(\mathbb{I}_n)$. All the Bockstein operations are natural by functoriality of $K$-theory.

The following commutative diagram describes how the inducing $*$-homomorphisms interact on $C^*$-algebra level.

$$\begin{tikzcd} SM_n \otimes A \arrow[r, "\xi"] \arrow[d, "\mu_n \otimes \mathrm{id}_A"] & SM_{nm} \otimes A \arrow[r, "\eta"] \arrow[d, "\mu_{nm} \otimes \mathrm{id}_A"] & M_m(SM_n) \otimes A \arrow[d, "\mu_n \otimes \mathrm{id}_{M_m} \otimes \mathrm{id}_A"] \\ \mathbb{I}_n \otimes A \arrow[r, "\kappa_{nm,n} \otimes \mathrm{id}_A"] \arrow[d, "\nu_n \otimes \mathrm{id}_A"] & \mathbb{I}_{nm} \otimes A \arrow[r, "\kappa_{n,nm} \otimes \mathrm{id}_A"] \arrow[d, "\nu_{nm} \otimes \mathrm{id}_A"] & M_m(\mathbb{I}_n) \otimes A \arrow[d, "\nu_n \otimes \mathrm{id}_{M_m} \otimes \mathrm{id}_A"] \\ A \arrow[r, equal] & A \arrow[r, "\mathrm{diag}_m^A"] & M_m(A) \end{tikzcd}$$

where $\xi = (\kappa_{nm,n} \otimes \mathrm{id}_A)|_{SM_n \otimes A}$ and $\eta = (\kappa_{n,nm} \otimes \mathrm{id}_A)|_{SM_{nm} \otimes A}$. Since we already know that the boundary maps are natural and explicitly either $\times n$ or $\times nm$, then we can use the above diagram to create a commutative diagram displaying the interactions between the Bockstein operations. For $i \in \mathbb{Z}/2$ we have

$$\begin{tikzcd}
K_i(A) \arrow[r, equal] \arrow[d, "\times n"] & K_i(A) \arrow[r, "\times m"] \arrow[d, "\times nm"] & K_i(A) \arrow[d, "\times n"] \\
K_i(A) \arrow[r, "\times m"] \arrow[d, "\mu_{i,A}^{(n)}"] & K_i(A) \arrow[r, equal] \arrow[d, "\mu_{i,A}^{(nm)}"] & K_i(A) \arrow[d, "\mu_{i,A}^{(n)}"] \\
K_i(A;\mathbb{Z}/n) \arrow[r, "\kappa_{i,A}^{(nm,n)}"] \arrow[d, "\nu_{i,A}^{(n)}"] & K_i(A;\mathbb{Z}/nm) \arrow[r, "\kappa_{i,A}^{(n,nm)}"] \arrow[d, "\nu_{i,A}^{(nm)}"] & K_i(A;\mathbb{Z}/n) \arrow[d, "\nu_{i,A}^{(n)}"] \\
K_{1-i}(A) \arrow[r, equal] \arrow[d, "\times n"] & K_{1-i}(A) \arrow[r, "\times m"] \arrow[d, "\times nm"] & K_{1-i}(A) \arrow[d, "\times n"] \\
K_{1-i}(A) \arrow[r, "\times m"] & K_{1-i}(A) \arrow[r, equal] & K_{1-i}(A)
\end{tikzcd}$$

which is commutative with exact columns.

## 6 A Description of the Künneth Short Exact Sequence

This section is devoted to finally describing the map $\beta$. We already have the description of $\alpha$, so this will complete our description of the Künneth short exact sequence for $A \in \mathcal{N}_{\text{min/max}}$. The goal of this chapter is to define a map $\theta$, which will ultimately be shown to be an isomorphism with its inverse implicitly being $\beta$. The large and technical Lemma 6.8 is required before we can finally prove, in Theorem 6.9, that $\theta$ is indeed the implicit inverse of $\beta$.
We will need the following lemma.

**Lemma 6.1.** *Let $A$ be a $C^*$-algebra, then the $*$-homomorphism $\eta : SA \to SA$ given by $\eta(f)(t) = f(1-t)$ has $K_i(\eta) = -\mathrm{id}_{K_i(SA)}$ for $i \in \mathbb{Z}/2$.*

*Proof.* Let $i \in \mathbb{Z}/2$. It is enough to show that the map $\xi : SA \to M_2(SA)$ given by

$$f \mapsto \begin{pmatrix} f & 0 \\ 0 & \eta(f) \end{pmatrix}$$

is homotopic to the zero map. This is because if the above is homotopic to the zero map, the we have $x + K_i(\eta)(x) = 0$ for all $x \in K_i(SA)$ and this can be seen by using the Morita isomorphism in $K$-theory $K_i(M_n(SA)) \to K_i(SA)$. Observe that the identity on $SA$ is homotopic to the following two $SA \to SA$ $*$-homomorphisms

$$f \mapsto \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ 0 & t \in [\frac{1}{2}, 1] \end{cases} \qquad f \mapsto \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ f(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

This means that $\xi$ is homotopic to $\Phi : SA \to M_2(SA)$ given by

$$f \mapsto \begin{cases} \begin{pmatrix} f(2t) & 0 \\ 0 & 0 \end{pmatrix} & t \in [0, \frac{1}{2}] \\ \begin{pmatrix} 0 & 0 \\ 0 & f(2-2t) \end{pmatrix} & t \in [\frac{1}{2}, 1] \end{cases}$$

A new homotopy $SA \to M_2(SA)$ for $s \in [0, \frac{1}{2}]$ is given by

$$f \mapsto \begin{cases} \begin{pmatrix} f(2t) & 0 \\ 0 & 0 \end{pmatrix} & t \in [0, \frac{1}{2}] \\ \begin{pmatrix} \cos(\pi s) & \sin(\pi s) \\ -\sin(\pi s) & \cos(\pi s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & f(2-2t) \end{pmatrix} \begin{pmatrix} \cos(\pi s) & -\sin(\pi s) \\ \sin(\pi s) & \cos(\pi s) \end{pmatrix} & t \in [\frac{1}{2}, 1] \end{cases}$$

The above is equal to $\Phi$ for $s = 0$ and for $s = \frac{1}{2}$ we have

$$f \mapsto \begin{cases} \begin{pmatrix} f(2t) & 0 \\ 0 & 0 \end{pmatrix} & t \in [0, \frac{1}{2}] \\ \begin{pmatrix} f(2-2t) & 0 \\ 0 & 0 \end{pmatrix} & t \in [\frac{1}{2}, 1] \end{cases} \tag{$\star$}$$

which is then homotopic to $\Phi$. A new homotopy $SA \to M_2(SA)$ for $s \in [0, 1]$ is given by

$$f \mapsto \begin{cases} \begin{pmatrix} f(2ts) & 0 \\ 0 & 0 \end{pmatrix} & t \in [0, \frac{1}{2}] \\ \begin{pmatrix} f(2s-2ts) & 0 \\ 0 & 0 \end{pmatrix} & t \in [\frac{1}{2}, 1] \end{cases}$$

and this is continuous for each $s \in [0, 1]$ since if $t = \frac{1}{2}$, then the segments agree. So we have a $*$-homomorphism $SA \to M_2(SA)$ for each $s \in [0, 1]$ and $s = 1$ gives ($\star$) and $s = 0$ gives the zero map. In conclusion, $\Phi$ is homotopic to the zero map and thus $\xi$ is homotopic to the zero map. $\square$

The following $*$-homomorphism will be the foundation for constructing the isomorphism $\theta$.

**Definition 6.2.** For $n \geq 2$ define a $*$-homomorphism $\omega^{(n)} : \mathbb{I}_n \otimes \mathbb{I}_n \to SM_n$ given by

$$\omega^{(n)}(f \otimes g)(t) = \begin{cases} f(2t)\nu_n(g) & \text{for } t \in [0, \frac{1}{2}] \\ \nu_n(f)g(2(1-t)) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

Note that $\mathbb{I}_n$ is nuclear so this map is not dependent on choosing a tensor norm.

The idea behind this map is to normalize each of the two functions in $\mathbb{I}_n$ before connecting them in $\mathbb{C}1_{M_n}$. This connection is done by reversing the direction of the function in the second argument and then gluing together on the matching endpoints. It is straight forward to check that $\omega^{(n)}$ is a $*$-homomorphism. For linearity, let $f, g \in \mathbb{I}_n$ $\lambda \in \mathbb{C}$, we then have

$$\omega^{(n)}(f \otimes g + \lambda(f \otimes g')) = \omega^{(n)}(f \otimes (g + \lambda g'))$$
$$= \begin{cases} f(2t)\nu_n(g + \lambda g') & \text{for } t \in [0, \frac{1}{2}] \\ \nu_n(f)(g(2(1-t)) + \lambda g'(2(1-t))) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$
$$= \omega^{(n)}(f \otimes g) + \lambda \omega^{(n)}(f \otimes g')$$

since $\nu_n(g + \lambda g') = \nu_n(g) + \lambda \nu_n(g')$. We also have the multiplication and involution structure preserved since $\nu_n(fg) = \nu_n(f)\nu_n(g)$ and $\nu_n(\overline{f}) = \overline{\nu_n(f)}$.

Another homomorphism can be constructed.

**Definition 6.3.** Let $A$ and $B$ be two $C^*$-algebras, $n \geq 2$, $i, j \in \mathbb{Z}/2\mathbb{Z}$, and let $\gamma = \min/\max$ denote the $C^*$-norm on the tensor product. We define the homomorphism

$$\Theta^{(n)_{A,B}}_{i,j;\gamma} : K_i(A; \mathbb{Z}/n) \otimes K_j(B; \mathbb{Z}/n) \to K_{i+j+1}(A \otimes_\gamma B)$$

by $x \otimes y \mapsto b \circ K_{i+j}(\mathrm{id}_A \otimes \omega^{(n)} \otimes \mathrm{id}_B) \circ \alpha_\gamma(x \otimes y)$ where $b$ is the Bott periodicity isomorphism and $\alpha_\gamma$ is the Künneth map.

$\Theta^{(n)}_{i,j;\gamma}$ is natural since $\alpha$ and $\beta$ are natural and since $K_{i+j}(A \otimes_\gamma \mathbb{I}_n \otimes_\gamma \mathbb{I}_n \otimes_\gamma B) \to K_{i+j}(A \otimes_\gamma SM_n \otimes_\gamma B)$ is natural in $A, B$ by functoriality of $K$-theory. We may just write $\Theta^{(n)}_{i,j;\gamma}$ when the associated $C^*$-algebras are stated in context.

**Lemma 6.4.** *Let $A$ and $B$ be $C^*$-algebras and let $\gamma = \min/\max$. Then the Bockstein operations are compatible with $\Theta$ in the following ways.*

1. *If $x \in K_i(A)$ and $y \in K_j(B; \mathbb{Z}/n)$, then*
$$\Theta^{(n)_{A,B}}_{i,j,\gamma}(\mu^{(n)}_{i,A}(x) \otimes y) = \alpha^{A,B}_\gamma(x \otimes \nu^{(n)}_{j,B}(y))$$

2. *If $x \in K_i(A; \mathbb{Z}/n)$ and $y \in K_j(B)$, then*
$$\Theta^{(n)_{A,B}}_{i,j,\gamma}(x \otimes \mu^{(n)}_{j,B}(y)) = -\alpha^{A,B}_\gamma(\nu^{(n)}_{i,B}(x) \otimes y)$$

3. *If $x \in K_i(A; \mathbb{Z}/n)$ and $y \in K_j(B; \mathbb{Z}/nm)$, then*
$$\Theta^{(nm)_{A,B}}_{i,j,\gamma}(\kappa^{(nm,n)}_{i,A}(x) \otimes y) = \Theta^{(n)_{A,B}}_{i,j,\gamma}(x \otimes \kappa^{(n,nm)}_{j,B}(y))$$

*4. If $x \in K_i(A;\mathbb{Z}/nm)$ and $y \in K_j(B;\mathbb{Z}/n)$, then*

$$\Theta^{(n)_{A,B}}_{i,j,\gamma}(\kappa^{(n,nm)}_{i,A}(x) \otimes y) = \Theta^{(nm)_{A,B}}_{i,j,\gamma}(x \otimes \kappa^{(nm,n)}_{j,B}(y))$$

*Proof.* We will omit notating which tensor norm is on the tensor products, we will just write $\otimes$ instead of $\otimes_\gamma$. The proof works for both min/max but it has no arguments involving this notion, so we choose .

*1.* For the sake of the having clean diagrams, we will use $K_{1-i}(A \otimes SM_n)$ instead of the isomorphic $K_i(A)$. Observe the following diagram

$$\begin{tikzcd}
K_{1-i}(A \otimes SM_n) \otimes K_{1-j}(\mathbb{I}_n \otimes B) \arrow[r, "\alpha"] \arrow[d, "\mu^{(n)}_{i,A}\otimes\mathrm{id}"] \arrow[dd, bend right=60, "\mathrm{id}\otimes\nu^{(n)}_{j,B}"'] & K_{i+j}(A \otimes SM_n \otimes \mathbb{I}_n \otimes B) \arrow[d, "K_{i+j}(\mathrm{id}_A\otimes\mu_n\otimes\mathrm{id}_{\mathbb{I}_n}\otimes\mathrm{id}_B)"] \arrow[dd, bend left=60, "\star"] \\
K_{1-i}(A \otimes \mathbb{I}_n) \otimes K_{1-j}(\mathbb{I}_n \otimes B) \arrow[r, "\alpha"] & K_{i+j}(A \otimes \mathbb{I}_n \otimes \mathbb{I}_n \otimes B) \arrow[d, "K_{i+j}(\mathrm{id}_A\otimes\omega^{(n)}\otimes\mathrm{id}_B)"] \\
K_{1-i}(A \otimes SM_n) \otimes K_{1-j}(B) \arrow[r, "\alpha"] & K_{i+j}(A \otimes SM_n \otimes B)
\end{tikzcd}$$

where $\star = K_{i+j}(\mathrm{id}_A \otimes \mathrm{id}_{SM_n} \otimes \nu_n \otimes \mathrm{id}_B)$. To prove the equality stated in *1.* we need to prove that the LHS of *1.* which is embedded as a the diagonal stair going from top left to bottom right commutes with RHS of *1.* which is embedded in the diagram as the top left going down to bottom left and then to bottom right. There are two naturality squares coming from the Künneth map $\alpha$, one is the big outer square and the other is the inner square. For the whole diagram to commute we will only need to prove that the right column commutes with $\star$. The diagram of interest is the following

$$\begin{tikzcd}
SM_n \otimes \mathbb{I}_n \arrow[r, "\mu_n\otimes\mathrm{id}_{\mathbb{I}_n}"'] \arrow[rr, bend left, "\mathrm{id}_{SM_n}\otimes\nu_n"] & \mathbb{I}_n \otimes \mathbb{I}_n \arrow[r, "\omega^{(n)}"'] & SM_n
\end{tikzcd} \tag{$\star$}$$

The bottom composition maps $f \otimes g \in SM_n \otimes \mathbb{I}_n$ in the following way

$$f \otimes g \mapsto \begin{cases} f(2t)\nu_n(g) & t \in [0, \frac{1}{2}] \\ \nu_n(f)g(2(1-t)) = 0 & t \in [\frac{1}{2}, 1] \end{cases} \in SM_n$$

while the top maps $f \otimes g \mapsto f\nu_n(g) \in SM_n$. The top and bottom of the diagram is homotopic by the homotopy $\varphi : SM_n \otimes \mathbb{I}_n \times [0,1] \to SM_n$ defined by

$$\varphi_h(f \otimes g) := \begin{cases} f\left(\frac{2}{1+h}t\right)\nu_n(g) & t \in [0, \frac{1+h}{2}] \\ 0 & t \in [\frac{1+h}{2}, 1] \end{cases}$$

$$\text{where} \quad \varphi_0(f \otimes g) = \omega^{(n)} \circ \mu \otimes \mathrm{id}_{\mathbb{I}_n}(f \otimes g) \quad \text{and} \quad \varphi_1(f \otimes g) = f\nu_n(g)$$

If we tensorize every $C^*$-algebra in $(\star)$ with $A$, $B$ and the maps with $\mathrm{id}_A$, $\mathrm{id}_B$, then we still have homotopic maps from $A \otimes SM_n \otimes \mathbb{I}_n \otimes B$ to $A \otimes SM_n \otimes B$. This means that $K$-theory of this new diagram commutes and this is precisely the right column in the big diagram. So we have

$$\Theta^{(n)_{A,B}}_{i,j,\gamma}(\mu^{(n)}_{i,A}(x) \otimes y) = \alpha^{A,B}_{\gamma}(x \otimes \nu^{(n)}_{j,B}(y))$$

for $x \in K_i(A)$ and $y \in K_j(B; \mathbb{Z}/n)$.

*2.* We will use the same argument as in the proof of *1.*. The statement of *2.* is equivalent to the following diagram commuting.

$$\begin{tikzcd}
K_{1-i}(A \otimes \mathbb{I}_n) \otimes K_{1-j}(SM_n \otimes B) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes \mu^{(n)}_{j,B}"] \arrow[dd, bend right=60, "\nu^{(n)}_{i,A} \otimes \mathrm{id}"'] & K_{i+j}(A \otimes \mathbb{I}_n \otimes SM_n \otimes B) \arrow[d, "K_{i+j}(\mathrm{id}_A \otimes \mathrm{id}_{\mathbb{I}_n} \otimes \mu_n \otimes \mathrm{id}_B)"] \arrow[dd, bend left=60, "\star"] \\
K_{1-i}(A \otimes \mathbb{I}_n) \otimes K_{1-j}(\mathbb{I}_n \otimes B) \arrow[r, "\alpha"] & K_{i+j}(A \otimes \mathbb{I}_n \otimes \mathbb{I}_n \otimes B) \arrow[d, "K_{i+j}(\mathrm{id}_A \otimes \omega^{(n)} \otimes \mathrm{id}_B)"] \\
K_{1-i}(A) \otimes K_{1-j}(SM_n \otimes B) \arrow[r, "\alpha"] & K_{i+j}(A \otimes SM_n \otimes B)
\end{tikzcd}$$

Naturality of $\alpha$ ensures that everything but the right column commutes. Just as in the proof of *1.*, we want to investigate what the diagram

$$\begin{tikzcd}
\mathbb{I}_n \otimes SM_n \arrow[r, "\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n"'] \arrow[rr, bend left, "\nu_n \otimes \mathrm{id}_{SM_n}"] & \mathbb{I}_n \otimes \mathbb{I}_n \arrow[r, "\omega^{(n)}"'] & SM_n
\end{tikzcd}$$

is doing in $K$-theory. The bottom composition maps $f \otimes g \in \mathbb{I}_n \otimes SM_n$ in the following way

$$f \otimes g \mapsto \begin{cases} f(2t)\nu_n(g) = 0 & t \in [0, \frac{1}{2}] \\ \nu_n(f) g(2(1-t)) & t \in [\frac{1}{2}, 1] \end{cases} \in SM_n$$

while the top maps $f \otimes g \mapsto g\nu_n(f)$. So it turns out that the diagram does not commute and the bottom composition flips the argument. We can prove that the bottom composition is homotopic with $f \otimes g \mapsto g\nu_n(f)$ with the argument flipped. To do this we define a homotopy, so for each $h \in [0, \frac{1}{2}]$ we define a $*$-homomorphism $\mathbb{I}_n \otimes SM_n \to SM_n$ by

$$f \otimes g \mapsto \begin{cases} 0 & t \in [0, h] \\ \nu_n(f) g(\frac{1-t}{1-h}) & t \in [h, 1] \end{cases}$$

The bottom composition of the diagram is given when $h = \frac{1}{2}$ and the map $f \otimes g \mapsto \overline{g}\nu_n(f)$ where $\overline{g} = (t \mapsto g(1-t))$ is given when $h = 0$. Lemma 6.1 states that inverting the argument of $SM_n$ flips the sign in $K$-theory. So we have that

$$K_{i+j}(\omega^{(n)} \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)) = -K_{i+j}(\nu_n \otimes \mathrm{id}_{SM_n})$$

When we tensorize with $A, B$ and $\mathrm{id}_A$,$\mathrm{id}_B$ on the maps, then we have gives

$$K_{i+j}(\mathrm{id}_A \otimes (\omega^{(n)} \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \mu_n)) \otimes \mathrm{id}_B) = -K_{i+j}(\mathrm{id}_A \otimes \nu_n \otimes \mathrm{id}_{SM_n} \otimes \mathrm{id}_B)$$

It can now be read from the big diagram that if $x \in K_i(A; \mathbb{Z}/n)$ and $y \in K_j(B)$, then

$$\Theta^{(n)_{A,B}}_{i,j,\gamma}(x \otimes \mu^{(n)}_{j,B}(y)) = -\alpha^{A,B}_\gamma(\nu^{(n)}_{i,B}(x) \otimes y)$$

*3.* We will first prove that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{I}_n \otimes \mathbb{I}_{nm} \arrow[r, "\mathrm{id}_{\mathbb{I}_n} \otimes \kappa_{n,nm}"] \arrow[d, "\kappa_{nm,n} \otimes \mathrm{id}_{\mathbb{I}_{nm}}"'] & \mathbb{I}_n \otimes \mathbb{I}_n \otimes M_m \arrow[d, "\omega^{(n)} \otimes \mathrm{id}_{M_m}"] \\
\mathbb{I}_{nm} \otimes \mathbb{I}_{nm} \arrow[r, "\omega^{(nm)}"] & SM_{nm}
\end{tikzcd}$$

In the clockwise direction we have

$$f \otimes g \overset{\mathrm{id}_{\mathbb{I}_n} \otimes \kappa_{n,nm}}{\mapsto} f \otimes \left( \sum_{i,j=1}^m g_{ij} \otimes E_{ij} \right) = \sum_{i,j=1}^m (f \otimes g_{ij} \otimes E_{ij})$$

where $E_{ij} \in M_m$ is the matrix with 1 in the $(i,j)$ position and 0 elsewhere, and $g_{ij} \in \mathbb{I}_n$ are the entries in $g \in M_m(\mathbb{I}_n) \cong \mathbb{I}_n \otimes M_m$. Now it is easy to see that $\omega^{(n)} \otimes \mathrm{id}_{M_m}$ maps this element to

$$\sum_{i,j=1}^m \omega^{(n)}(f \otimes g_{ij}) \otimes E_{ij} \in SM_{nm}$$

For the counter clockwise direction, we have $f \otimes g \overset{\kappa_{nm,n} \otimes \mathrm{id}_{\mathbb{I}_{nm}}}{\mapsto} (f \otimes 1_{M_m}) \otimes g$. Think of $f \otimes 1_{M_m}$ and $g$ as elements in $M_m(M_n)$, then the final map is

$$\omega^{(nm)}((f \otimes 1_{M_m}) \otimes g) = \begin{cases} \nu_{nm}(g)(f \otimes 1_{M_m})(2t) & t \in [0, \frac{1}{2}] \\ \nu_n(f) g(2(1-t)) & t \in [\frac{1}{2}, 1] \end{cases} \in M_m(SM_n) = SM_{mn}$$

It is not immediate to see that the diagram commutes, and to make it clear, we can compare each $n \times n$ block in $M_m(SM_n) \cong SM_{nm}$. So let $i, j \in \{1, ..., m\}$ with $i \neq j$, then we see that

$$(\omega^{(nm)} \circ (\kappa_{nm,n} \otimes \mathrm{id}_{\mathbb{I}_{nm}})(f \otimes g))_{ij} = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ \nu_n(f) g_{ij}(2(1-t)) & t \in [\frac{1}{2}, 1] \end{cases}$$

$$((\omega^{(n)} \otimes \mathrm{id}_{M_m}) \circ (\mathrm{id}_{\mathbb{I}_n} \otimes \kappa_{n,nm})(f \otimes g))_{ij} = \begin{cases} \nu_{nm}(g_{ij}) f(2t) = 0 & t \in [0, \frac{1}{2}] \\ \nu_n(f) g_{ij}(2(1-t)) & t \in [\frac{1}{2}, 1] \end{cases}$$

and for $i = j$ we have both equal to $\omega^{(n)}(f \otimes g_{ij})$. So the diagram commutes and likewise in $K$-theory. This gives us the commuting diagram

$$\begin{tikzcd}
K_*(A \otimes \mathbb{I}_n \otimes \mathbb{I}_{nm} \otimes B) \arrow[r, "K_*(\mathrm{id}_A \otimes \mathrm{id}_{\mathbb{I}_n} \otimes \kappa_{n,nm} \otimes \mathrm{id}_B)"] \arrow[d, "K_*(\mathrm{id}_A \otimes \kappa_{nm,n} \otimes \mathrm{id}_{\mathbb{I}_{nm}} \otimes \mathrm{id}_B)"'] & K_*(A \otimes \mathbb{I}_n \otimes \mathbb{I}_n \otimes M_m \otimes B) \arrow[d, dashed, "K_*(\mathrm{id}_A \otimes \omega^{(n)} \otimes \mathrm{id}_{M_n} \otimes \mathrm{id}_B)"] \\
K_*(A \otimes \mathbb{I}_{nm} \otimes \mathbb{I}_{nm} \otimes B) \arrow[r, "K_*(\mathrm{id}_A \otimes \omega^{(nm)} \otimes \mathrm{id}_B)"] & K_*(A \otimes SM_{nm} \otimes B)
\end{tikzcd} \tag{$\diamond$}$$

With this diagram in mind, consider the following commuting diagram

$$\begin{tikzcd}
K_{1-i}(A \otimes \mathbb{I}_n) \otimes K_{1-j}(\mathbb{I}_{nm} \otimes B) \arrow[r, "\mathrm{id} \otimes \kappa_{j,B}^{(n,nm)}"] \arrow[d, "\alpha"] \arrow[dd, bend right, "\star"'] & K_{1-i}(A \otimes \mathbb{I}_n) \otimes K_{1-j}(\mathbb{I}_n \otimes M_m \otimes B) \arrow[d, "\alpha"] \\
K_{i+j}(A \otimes \mathbb{I}_n \otimes \mathbb{I}_{nm} \otimes B) \arrow[r, "K_{i+j}(\mathrm{id}_A \otimes \mathrm{id}_{\mathbb{I}_n} \otimes \kappa_{n,nm} \otimes \mathrm{id}_B)"] \arrow[dr, "K_{i+j}(\mathrm{id}_A \otimes \kappa_{nm,n} \otimes \mathrm{id}_{\mathbb{I}_{nm}} \otimes \mathrm{id}_B)"] & K_{i+j}(A \otimes \mathbb{I}_n \otimes \mathbb{I}_n \otimes M_m \otimes B) \\
K_{1-i}(A \otimes \mathbb{I}_{nm}) \otimes K_{1-j}(\mathbb{I}_{nm} \otimes B) \arrow[r, "\alpha"] & K_{i+j}(A \otimes \mathbb{I}_{nm} \otimes \mathbb{I}_{nm} \otimes B)
\end{tikzcd}$$

where $\star = \kappa_{i,A}^{(nm,n)} \otimes \mathrm{id}$. This diagram commutes from naturality of $\alpha$ and composing this diagram with $(\diamond)$ gives us a new commuting diagram which traces out and verifies the statement of *3.* In other words, if $x \in K_i(A; \mathbb{Z}/n)$ and $y \in K_j(B; \mathbb{Z}/nm)$, then

$$\Theta_{i,j,\gamma}^{(nm)_{A,B}}(\kappa_{i,A}^{(nm,n)}(x) \otimes y) = \Theta_{i,j,\gamma}^{(n)_{A,B}}(x \otimes \kappa_{j,B}^{(n,nm)}(y))$$

*4.*

We have the exact same argument as in the proof of *3.*. It also uses commutativity of $\diamond$, but the diagram below it will needs to be adapted to our hypothesis. The diagram again commutes from naturality of $\alpha$ and this point is proven. $\square$

It is now time to define the important homomorphism $\theta$.

**Theorem 6.5.** *Let $\gamma = \min/\max$ be the choice of $C^*$-norm. For all $C^*$-algebras $A$ and $B$ there is a natural homomorphism*

$$\theta : \substack{\mathrm{Tor}(K_0(A), K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A), K_0(B))} \to \operatorname{coker}\left(\alpha : \substack{K_0(A) \otimes K_0(B) \\ \oplus \\ K_1(A) \otimes K_1(B)} \to K_0(A \otimes_\gamma B)\right)$$

*The homomorphism is given as follows: Let $i \in \mathbb{Z}/2$ decide the direct summand, then for $(x, n, y) \in \mathrm{Tor}(K_i(A), K_{1-i}(B))$ where $n \geq 2$, $x \in K_i(A)[n]$ and $y \in K_{1-i}(B)[n]$, we have*

$$\theta(x, n, y) = [\Theta_{1-i,i;\gamma}^{(n)_{A,B}}(\tilde{x} \otimes \tilde{y})]$$

*where $\tilde{x} \in K_{1-i}(A; \mathbb{Z}/n)$ and $\tilde{y} \in K_i(B; \mathbb{Z}/n)$ are any elements satisfying $\nu_{1-i,A}^{(n)}(\tilde{x}) = x$ and $\nu_{i,B}^{(n)}(\tilde{y}) = y$.*

**Remark 6.6.** This theorem also covers the $K_1$ case since if $(A, SB)$ is chosen instead of $(A, B)$, then we get the natural homomorphism

$$\theta : \begin{smallmatrix}\mathrm{Tor}(K_0(A),K_0(B))\\ \oplus \\ \mathrm{Tor}(K_1(A),K_1(B))\end{smallmatrix} \to \operatorname{coker}\left(\alpha : \begin{smallmatrix}K_0(A)\otimes K_1(B)\\ \oplus \\ K_1(A)\otimes K_0(B)\end{smallmatrix} \to K_1(A \otimes_\gamma B)\right)$$

which is the other part of the the map

$$\mathrm{Tor}(K_*(A), K_*(B)) \to \operatorname{coker}(\alpha_\gamma^{A,B})$$

using the notation from Definition 4.1.

**Definition 6.7.** Let $A, B$ be $C^*$-algebras and $\gamma = \min/\max$ be the $C^*$-norm on the tensor product. We define

$$\theta_\gamma^{A,B} : \mathrm{Tor}(K_*(A), K_*(B)) \to \operatorname{coker}(\alpha_\gamma^{A,B})$$

to be the full homomorphism incorporating both the homomorphisms from Theorem 6.9 and Remark 6.6.

*Proof of Theorem 6.5.* The exact sequence $K_{1-i}(A;\mathbb{Z}/n) \xrightarrow{\nu_{1-i,A}^{(n)}} K_i(A) \xrightarrow{\times n} K_i(A)$ from the six-term sequence defining the Bockstein operations, tells us that the map $\nu_{1-i,A}^{(n)}$ exactly hits all of $K_i(A)[n]$. So $\theta$ is well defined in that sense. Since there is a choice involved in finding $\tilde{x}$ and $\tilde{y}$, we need to make sure that it is irrelevant which $\tilde{x}$ and $\tilde{y}$ is chosen.

To this end, let $\tilde{x}, \tilde{x}' \in K_{1-i}(A;\mathbb{Z}/n)$ and $\tilde{y}, \tilde{y}' \in K_i(B;\mathbb{Z}/n)$ with $\nu_{1-i,A}^{(n)}(\tilde{x}) = x = \nu_{1-i,A}^{(n)}(\tilde{x}')$ and $\nu_{i,B}^{(n)}(\tilde{y}) = y = \nu_{i,B}^{(n)}(\tilde{y}')$. Observe that

$$[\Theta_{1-i,i;\gamma}^{(n)}(\tilde{x} \otimes \tilde{y})] = [\Theta_{1-i,i;\gamma}^{(n)}(\tilde{x}' \otimes \tilde{y}')]$$

is equivalent to $\Theta_{1-i,i;\gamma}^{(n)}(\tilde{x} \otimes \tilde{y}) - \Theta_{1-i,i;\gamma}^{(n)}(\tilde{x}' \otimes \tilde{y}')$ being in the image of $\alpha$.

We know that $\tilde{x} \otimes \tilde{y} - \tilde{x}' \otimes \tilde{y}' = (\tilde{x} - \tilde{x}') \otimes \tilde{y} + \tilde{x}' \otimes (\tilde{y} - \tilde{y}')$, and this gives

$$\begin{aligned}&\Theta_{1-i,i;\gamma}^{(n)}\left(\tilde{x} \otimes \tilde{y}\right) - \Theta_{1-i,i;\gamma}^{(n)}(\tilde{x}' \otimes \tilde{y}'\Big) \\ &= \Theta_{1-i,i;\gamma}^{(n)}\left((\tilde{x} - \tilde{x}') \otimes \tilde{y}\right) + \Theta_{1-i,i;\gamma}^{(n)}\left(\tilde{x}' \otimes (\tilde{y} - \tilde{y}')\right)\end{aligned}$$

Since we have $\nu_{1-i,A}^{(n)}(\tilde{x} - \tilde{x}') = 0$ and $\nu_{i,B}^{(n)}(\tilde{y} - \tilde{y}') = 0$, then exactness ensures that we have

$$= \Theta_{1-i,i;\gamma}^{(n)}\left(\mu_{1-i,A}^{(n)}(z_1) \otimes \tilde{y}\right) + \Theta_{1-i,i;\gamma}^{(n)}\left(\tilde{x}' \otimes \mu_{i,B}^{(n)}(z_2)\right)$$

for some $z_1 \in K_{1-i}(A)$ and $z_2 \in K_i(B)$. Now use 1. and 2. from Lemma 6.4 to see that

$$\begin{aligned}&= \alpha\left(z_1 \otimes \nu_{i,B}^{(n)}(\tilde{y})\right) - \alpha\left(\nu_{1-i,A}^{(n)}(\tilde{x}') \otimes z_2\right) \\ &= \alpha\left(z_1 \otimes \nu_{i,B}^{(n)}(\tilde{y}) - \nu_{1-i,A}^{(n)}(\tilde{x}')\right)\end{aligned}$$

which is easily seen to be in the image of $\alpha$. This means that the choice of $\tilde{x}$ and $\tilde{y}$ is ours to make and that it has no effect on the definition of $\theta$.

Now we know that $\theta$ is well defined on the generators of Tor, but we also need this map to respect the relations on these generators for it to be a homomorphism. These relations are the four torsion product relations from Definition 2.2. In this context we need to prove the following:

*1.*

$$\theta(x,n,y)+\theta(x,n,y') \;=\; \theta(x,n,y+y') \qquad (x\in K_i(A)[n],\ y,y'\in K_{1-i}(B)[n])$$

Let $x\in K_i(A)[n]$ and $y,y'\in K_{1-i}(B)[n]$, then we can find $\tilde{x}\in K_{1-i}(A;\mathbb{Z}/n)$ and $\tilde{y},\tilde{y}'\in K_i(B;\mathbb{Z}/n)$ as in the theorem such that

$$\begin{aligned}\theta(x,n,y)+\theta(x,n,y') &= [\Theta^{(n)}_{1-i,i,\gamma}(\tilde{x}\otimes\tilde{y})]+[\Theta^{(n)}_{1-i,i,\gamma}(\tilde{x}\otimes\tilde{y}')]\\ &= [\Theta^{(n)}_{1-i,i,\gamma}(\tilde{x}\otimes(\tilde{y}+\tilde{y}'))]\overset{\star}{=}\theta(x,n,y+y')\end{aligned}$$

where $\star$ is justified since we can choose $\tilde{y}+\tilde{y}'\in K_i(B;\mathbb{Z}/n)$ such that $\nu^{(n)}_{i,B}(\tilde{y}+\tilde{y}')=y+y'$.

*2.*

$$\theta(x,n,y)+\theta(x',n,y) \;=\; \theta(x+x',n,y) \qquad (x,x'\in K_i(A)[n],\ y\in K_{1-i}(B)[n])$$

This is done by using the exact same method as in the proof of *1.*

*3.*

$$\theta(mx,n,y) \;=\; \theta(x,nm,y) \qquad (x\in K_i(A)[nm],\ y\in K_{1-i}(B)[n])$$

Let $\tilde{x}\in K_{1-i}(A;\mathbb{Z}/nm)$ and $\tilde{y}\in K_i(B;\mathbb{Z}/n)$ as in the statement of the theorem. The element $\kappa^{(nm,n)}_{i,B}(\tilde{y})\in K_i(B;\mathbb{Z}/nm)$ has $\nu^{(nm)}_{i,B}(\kappa^{(nm,n)}_{i,B}(\tilde{y}))=y=\nu^{(n)}_{i,B}(\tilde{y})$ since $\nu^{(nm)}_{i,B}\circ\kappa^{(nm,n)}_{i,B}=\nu^{(n)}_{i,B}$ as can be seen under Definition 5.3 in the diagram describing the Bockstein interactions. This means that we have

$$\theta(x,nm,y)=[\Theta^{(nm)}_{1-i,i,\gamma}(\tilde{x}\otimes\kappa^{(nm,n)}_{i,B}(\tilde{y}))]$$

In the same spirit, the element $\kappa^{(n,nm)}_{1-i,A}(\tilde{x})\in K_{1-i}(A;\mathbb{Z}/n)$ has $\nu^{(n)}_{1-i,A}(\kappa^{(n,nm)}_{1-i,A}(\tilde{x}))=mx=\nu^{(nm)}_{1-i,A}(\tilde{x})$ since $\nu^{(n)}_{1-i,A}\circ\kappa^{(n,nm)}_{1-i,A}=(\times m)\circ\nu^{(nm)}_{1-i,A}$ and this can also be seen in the same diagram under Definition 5.3. Similarly, we have

$$\theta(mx,n,y)=[\Theta^{(n)}_{1-i,i,\gamma}(\kappa^{(n,nm)}_{1-i,A}(\tilde{x})\otimes\tilde{y})]$$

Lemma 6.4 part 4. states that $\Theta^{(nm)}_{1-i,i,\gamma}(\tilde{x}\otimes\kappa^{(nm,n)}_{i,B}(\tilde{y}))=\Theta^{(n)}_{1-i,i,\gamma}(\kappa^{(n,nm)}_{1-i,A}(\tilde{x})\otimes\tilde{y})$ and this directly implies $\theta(mx,n,y)=\theta(x,nm,y)$.

*4.*
$$\theta(x,n,my) \quad = \quad \theta(x,nm,y) \qquad (x \in K_i(A)[n],\ y \in K_{1-i}(B)[nm])$$

This is done by using the exact same method as in the proof of *3.*, but this time we invoke Lemma 6.4 part 3.

Naturality of $\theta$ comes from the fact that $\Theta^{(n)}_{i,j,\gamma}$ is natural and from the fact that we can choose $\tilde{x}$ freely. We will show this in more detail. Let $\varphi : A \to C$ be a $*$-homomorphism, then naturality of the Künneth map ensures that the diagram

$$\begin{tikzcd}
(E :=) \begin{smallmatrix} K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B) \end{smallmatrix} \arrow[r, "\alpha^{A,B}"] \arrow[d, "K_0(\varphi)\otimes\mathrm{id}\oplus K_1(\varphi)\otimes\mathrm{id}"'] & K_0(A\otimes_\gamma B) \arrow[d, "K_0(\varphi\otimes\mathrm{id}_B)"] \\
(F :=) \begin{smallmatrix} K_0(C)\otimes K_0(B) \\ \oplus \\ K_1(C)\otimes K_1(B) \end{smallmatrix} \arrow[r, "\alpha^{C,B}"] & K_0(C\otimes_\gamma B)
\end{tikzcd}$$

is commutative. Commutativity now ensures that $K_0(\varphi\otimes\mathrm{id}_B)\circ\alpha_1(E) \subseteq \alpha_2(F)$ which means that $\pi : K_0(A\otimes_\gamma B)/\alpha_1(E) \to K_0(C\otimes_\gamma B)/\alpha_2(F)$ is well defined by $a+\alpha_1(E) \mapsto K_0(\varphi\otimes\mathrm{id}_B)(a)+\alpha_2(F)$. The map $\pi : \mathrm{coker}(\alpha_1) \to \mathrm{coker}(\alpha_2)$ is canonical and fits into the diagram

$$\begin{tikzcd}
\begin{smallmatrix} \mathrm{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(A),K_0(B)) \end{smallmatrix} \arrow[r, "\theta^{A,B}"] \arrow[d, "\psi"] & \mathrm{coker}(\alpha_1) \arrow[d, "\pi"] \\
\begin{smallmatrix} \mathrm{Tor}(K_0(C),K_1(B)) \\ \oplus \\ \mathrm{Tor}(K_1(C),K_0(B)) \end{smallmatrix} \arrow[r, "\theta^{C,B}"] & \mathrm{coker}(\alpha_2)
\end{tikzcd} \tag{$\star$}$$

where $\psi$ is the homomorphism $\mathrm{Tor}(K_0(\varphi),\mathrm{id}_{K_1(B)})\oplus\mathrm{Tor}(K_1(\varphi),\mathrm{id}_{K_0(B)})$. The point is to prove that this diagram commutes. To do this, let $i \in \mathbb{Z}/2$ be a choice of direct summand in the domain of $\psi$. Let $(x,n,y) \in \mathrm{Tor}(K_i(A),K_{1-i}(B))$, then the "solid" part of the next diagram explains how this element is mapped in the above diagram.

$$\begin{tikzcd}
(x,n,y) \arrow[r, mapsto, "\theta^{A,B}"] \arrow[d, mapsto, "\mathrm{Tor}(K_i(\varphi),\mathrm{id}_{K_{1-i}(B)})"] & {[\Theta^{(n)A,B}_{1-i,i,\gamma}(\tilde{x}\otimes\tilde{y})]} \arrow[d, mapsto, dashed, "\pi"] \\
(\varphi(x),n,y) \arrow[r, mapsto, "\theta^{C,B}"] & {[\Theta^{(n)C,B}_{1-i,i,\gamma}(\widetilde{\varphi(x)}\otimes\tilde{y})]}
\end{tikzcd} \tag{$\sharp$}$$

We remind the reader that $\widetilde{\varphi(x)} \in K_{1-i}(C;\mathbb{Z}/n)$ is such that $\nu^{(n)}_{1-i,C}(\widetilde{\varphi(x)}) = \varphi(x) \in K_i(C)$ as with $\tilde{x},\tilde{y}$ in their respective $K$ groups. The dotted line is what we need to prove. The following diagram commutes by functoriality of $K$-theory

$$\begin{tikzcd}
K_{1-i}(A;\mathbb{Z}/n) \arrow[r, "\nu^{(n)}_{1-i,A}"] \arrow[d, "K_{1-i}(\varphi\otimes\mathrm{id}_{\mathbb{I}_n})"'] & K_i(A) \arrow[d, "K_i(\varphi)"] \\
K_{1-i}(C;\mathbb{Z}/n) \arrow[r, "\nu^{(n)}_{1-i,C}"] & K_i(C)
\end{tikzcd}$$

So we can choose $K_{1-i}(\varphi \otimes \mathrm{id}_{\mathbb{I}_n})(\tilde{x}) \in K_{1-i}(C;\mathbb{Z}/n)$ in place of $\widetilde{\varphi(x)}$ and get

$$[\Theta^{(n)C,B}_{1-i,i,\gamma}(\widetilde{\varphi(x)} \otimes \tilde{y})] = [\Theta^{(n)C,B}_{1-i,i,\gamma}(K_{1-i}(\varphi \otimes \mathrm{id}_{\mathbb{I}_n})(\tilde{x}) \otimes \tilde{y})]$$

Now since $\Theta^{(n)}_{i,j,\gamma}$ is natural, then the following diagram commutes

$$\begin{tikzcd}
K_{1-i}(A;\mathbb{Z}/n) \otimes K_i(B;\mathbb{Z}/n) \arrow[r, "\Theta^{(n)A,B}_{1-i,i,\gamma}"] \arrow[d, "K_{1-i}(\varphi\otimes \mathrm{id}_{\mathbb{I}_n})\otimes \mathrm{id}_{K_i(B;\mathbb{Z}/n)}"'] & K_0(A \otimes_\gamma B) \arrow[d, "K_0(\varphi\otimes\mathrm{id}_B)"] \\
K_{1-i}(C;\mathbb{Z}/n) \otimes K_i(B;\mathbb{Z}/n) \arrow[r, "\Theta^{(n)C,B}_{1-i,i,\gamma}"] & K_0(C \otimes_\gamma B)
\end{tikzcd}$$

and since $\pi : \mathrm{coker}(\alpha_1) \to \mathrm{coker}(\alpha_2)$ is defined on representatives, then we have

$$[\Theta^{(n)}_{1-i,i,\gamma}(K_{1-i}(\varphi \otimes \mathrm{id}_{\mathbb{I}_n})(\tilde{x}) \otimes \tilde{y})] = \pi([\Theta^{(n)}_{1-i,i,\gamma}(\tilde{x} \otimes \tilde{y})])$$

This ensures that the dotted line in ($\sharp$) commutes with the diagram, and as a direct consequence we see that ($\star$) commutes. This proves that $\theta$ is natural. $\square$

This technical lemma will be the last thing we need before we can prove that $\theta$ is the inverse of $\beta$.

**Lemma 6.8.** *Let $P_0$ and $B$ be $C^*$-algebras and let $\psi : P_0 \to B$ be a $*$-homomorphism such that $K_*(\psi) : K_*(P_0) \to K_*(B)$ is surjective. Let $P_1$ be the mapping cone of $\psi$ such that we get the short exact sequence*

$$0 \longrightarrow SB \xrightarrow{\phi} P_1 \xrightarrow{\rho} P_0 \longrightarrow 0$$

*which, as seen in the proof of Theorem 4.5, gives us the short exact sequence*

$$0 \longrightarrow K_i(P_1) \xrightarrow{K_i(\rho)} K_i(P_0) \xrightarrow{K_i(\psi)} K_i(B) \longrightarrow 0$$

*for $i \in \mathbb{Z}/2\mathbb{Z}$. Then the map $\phi^{(n)}_i : K_{1-i}(B;\mathbb{Z}/n) \to K_i(P_1;\mathbb{Z}/n)$ can be computed as follows: Given $y \in K_{1-i}(B;\mathbb{Z}/n)$ let $p \in K_i(P_0)$ such that $\nu^{(n)}_{i,B}(y) = K_i(\psi)(p) \in K_i(B)$. Now there is a unique $z \in K_i(P_1)$ such that $K_i(\rho)(z) = np$ and the $*$-homomorphism is defined to be $\phi^{(n)}_i(y) = -\mu^{(n)}_{i,P_1}(z)$.*

*Proof.* The first thing to handle is to show that definition of $\phi^{(n)}_i$ is independent of the choice of $p \in K_i(P_0)$. Let $y \in K_{1-i}(B;\mathbb{Z}/n)$ and let $p_1, p_2 \in K_i(P_0)$ with $K_i(\psi)(p_1) = \nu^{(n)}_{1-i,B}(y) = K_i(\psi)(p_2)$. Now let $z_1, z_2 \in K_i(P_1)$ be such that $K_i(\rho)(z_1) = np_1$ and $K_i(\rho)(z_2) = np_2$. We have $u \in K_i(P_1)$ with $K_i(\rho)(u) = p_1 - p_2$ since $K_i(\psi)(p_1-p_2) = 0$. So we see that $K_i(\rho)(z_1-z_2-nu) = 0$ meaning $z_1-z_2 = nu$ from injectivity. Now this implies that $\mu^{(n)}_{i,P_1}(z_1 - z_2) = \mu^{(n)}_{i,P_1}(nu) = 0$ since the sequence $K_i(P_1) \xrightarrow{\times n} K_i(P_1) \xrightarrow{\mu^{(n)}_{i,P_1}} K_i(P_1;\mathbb{Z}/n)$ is exact, so $\mu^{(n)}_{i,P_1}(z_1) = \mu^{(n)}_{i,P_1}(z_2)$ and

$\phi_i^{(n)}$ is well defined as a map.
Now to see that $\phi_i^{(n)}$ is a homomorphism, let $x, y \in K_{1-i}(B; \mathbb{Z}/n)$. Then $\nu_{1-i,B}^{(n)}(x)$ and $\nu_{1-i,B}^{(n)}(y)$ lifts to $p_1$ and $p_2$ in $K_i(P_0)$ respectively, and we have $z_1, z_2 \in K_i(P_1)$ with $K_i(\rho)(z_1) = np_1$, $K_i(\rho)(z_2) = np_2$. This means that we can compute $\phi_i^{(n)}(x) + \phi_i^{(n)}(y) = -\mu_{i,P_1}^{(n)}(z_1) - \mu_{i,P_1}^{(n)}(z_2) = -\mu_{i,P_1}^{(n)}(z_1+z_2)$. Now observe that $K_i(\rho)(z_1+z_2) = n(p_1+p_2)$ with $K_i(\psi)(p_1+p_2) = \nu_{1-i,B}^{(n)}(x+y) = \nu_{1-i,B}^{(n)}(x) + \nu_{1-i,B}^{(n)}(y)$. This means that $\phi_i^{(n)}(x) + \phi_i^{(n)}(x) = \phi_i^{(n)}(x+y)$ which concludes the question about $\phi_i^{(n)}$ being a homomorphism.
As a preparation for proving that this homomorphisms is in fact the correct one, we will construct a diagram which shows an equivalent but more elegant process for the computation of $\phi_i^{(n)}$. Since we want to connect $y \in K_{1-i}(B; \mathbb{Z}/n)$ with $z \in P_1$, the following diagram will be of great use.

$$\begin{tikzcd}
0 \arrow[r] & SM_n(B) \arrow[r, "\mu_n \otimes \mathrm{id}_B"] & \mathbb{I}_n(B) \arrow[r, "\nu_n \otimes \mathrm{id}_B"] & B \arrow[r] & 0 \\
0 \arrow[r] & SM_n(B) \arrow[u, equal] \arrow[r] \arrow[d, equal] & D \arrow[u, "\eta"] \arrow[r] \arrow[d, "\xi"] & P_0 \arrow[u, "\psi"] \arrow[r] \arrow[d, "\mathrm{diag}_n"] & 0 \\
0 \arrow[r] & SM_n(B) \arrow[r, "\mathrm{id}_{M_n} \otimes \phi"] & M_n(P_1) \arrow[r, "\mathrm{id}_{M_n} \otimes \rho"] & M_n(P_0) \arrow[r] & 0
\end{tikzcd}$$

The top is the short exact sequence from the definition of $K_{1-i}(B; \mathbb{Z}/n)$ and the associated homomorphisms. The bottom is the mapping cone short exact sequence for $P_1$ tensorized with $M_n$. We define $D = \{(a, f) \in P_0 \oplus \mathbb{I}_n(B) \mid \psi(a) = (\nu_n \otimes \mathrm{id}_B)(f)\}$ to be the pull back of the top right with the obvious maps making the middle row exact. The vertical maps are defined by $\eta(a, f) = f$ and $\xi(a, f) = (1_{M_n} \otimes a, f)$ (so that entries both are $n$ squares), and we remind the reader that $P_1 = \{(a, f) \in P_0 \oplus CB \mid \psi(a) = f(1)\}$ where we view $\mathbb{I}_n(B) \subseteq M_n(CB)$. Now it can be seen that the whole diagram commutes.
We will show that $K_i(\eta) : K_i(D) \to K_i(\mathbb{I}_n(B))$ is surjective in K-theory. To do this, we set up an equivalent case in the following commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u, equal] \arrow[r, "k"] & D \arrow[u, "h"] \arrow[r, "s"] & E \arrow[u, "t"] \arrow[r] & 0
\end{tikzcd}$$

with exact rows and surjective $t$. If $b \in B$, then we know that we can find $d \in D$ such that $t \circ s(d) = g(b)$ and thus $g(b - h(d)) = 0$. We can find $a \in A$ with $f(a) = b - h(d)$ and this gives us $h(d + k(a)) = h(d) + h \circ k(a) = h(d) + f(a) = h(d) + b - h(d) = b$. So $h$ is surjective which proves that $K_*(\eta)$ is surjective.
Let $y \in K_i(B; \mathbb{Z}/n)$, then from the previous we have $d \in K_i(D)$ with $K_i(\eta)(d) = y$

and $K_i(\xi)(d) \in K_i(M_n(P_1))$. The $z \in K_i(P_1)$ as in the statement of the lemma, is given by $K_i(\nu_n \otimes \mathrm{id}_n)(y)$, lifted to $p \in K_i(P_0)$ (which we can choose so as to be hit by the element $d$ in the map $K_i(D) \to K_i(P_0)$), which is then sent to $K_i(M_n(P_0)) \cong K_i(P_0)$ by $K_i(\mathrm{diag}_n)$ which corresponds to $np \in K_i(P_0)$. Again, this $np$ can be lifted to $K_i(M_n(P_1))$ since $np \in K_i(P_0) \cong K_i(M_n(P_0))$ lifts uniquely to $z \in K_i(P_1) \cong K_i(M_n(P_1))$. Observe that since the above diagram commutes, as well as in $K$-theory, then we have $m(K_i(\xi)(d)) = z$ (where $m : K_i(M_n(P_1)) \to K_i(P_1)$ is the Morita isomorphism) and by extension $\mu^{(n)}_{i,P_1}(m(K_i(\xi)(d))) = \mu^{(n)}_{i,P_1}(z)$. This $z$ is chosen according to the prescribed process and, as we know, this ensures that $-\mu^{(n)}_{i,P_1}(z)$ is unique. This shows that $\phi^{(n)}_i$ can be computed with a simple lift in the diagram.

We still need to show that $\phi^{(n)}_i$ corresponds to the correct homomorphism. To this end, we have diagram

$$\begin{tikzcd} SD \arrow[r, "\overline{S}\eta"] \arrow[d, "S\xi"] & S\mathbb{I}_n(B) \cong \mathbb{I}_n(SB) \arrow[d, "\mathrm{id}_{\mathbb{I}_n} \otimes \phi"] \\ SM_n(P_1) \arrow[r, "\mu_n \otimes \mathrm{id}_{P_1}"] & \mathbb{I}_n(P_1) \end{tikzcd}$$

where $\overline{S}\eta(f)(s) = \eta(f(-s))$ for $f \in C_0((-1,1), D)$ (it flips the sign in K-theory, Lemma 6.1). We claim that this diagram commutes up to homotopy and therefore commutes in $K$-theory. If this diagram commutes in $K$-theory, then we see that $x \in K_{1-i}(B; \mathbb{Z}/n)$ mapped by $\phi^{(n)}_i : K_{1-i}(B; \mathbb{Z}/n) \to K_i(P_1; \mathbb{Z}/n)$ is the same as lifting $x$ to $K_{1-i}(D)$ and then using the map $K_i(D) \to K_i(M_n(P_1)) \to K_i(P_1; \mathbb{Z}/n)$, but we know that this last map agrees with the map described in the statement of this lemma. We will therefore use the rest of the proof to prove that this diagram commutes up to homotopy.

The isomorphism $S\mathbb{I}_n(B) \xrightarrow{\cong} \mathbb{I}_n(SB)$ sends $f(s,t) \mapsto f(t,s)$ and this can be seen directly from the following identifications

| | |
|---|---|
| $S(\mathbb{I}_n(B))$ is | $\mathbb{I}_n(SB)$ is |
| $f \in C_0((-1,1) \times [-1,1], M_n(B))$ | $f \in C_0([-1,1] \times (-1,1), M_n(B))$ |
| subject to | subject to |
| $f(-1,t) = 0,\ f(1,t) = 0,$ | $f(-1,s) = 0,\ f(1,s) \in 1_{M_n} \otimes B,$ |
| $f(s,-1) = 0,\ f(s,1) \in 1_{M_n} \otimes B$ | $f(t,-1) = 0,\ f(t,1) = 0$ |

The identifications are possible since $C_0(X, C_0(Y,A)) \cong C_0(X \times Y, A)$ in a canonical way. $\mathrm{id}_{\mathbb{I}_n} \otimes \phi$ is just the entrywise inclusion (in $M_n(P_1)$) of $SB$ into the $CB$ summand of $P_1$. For comprehensions sake, we will reiterate the following:

$$D = \{(a, f) \in P_0 \oplus \mathbb{I}_n(B) \mid \psi(a) = (\nu_n \otimes \mathrm{id}_B)(f)\}$$
$$P_1 = \{(a, f) \in P_0 \oplus CB \mid \psi(a) = f(1)\}$$

To make everything a lot more clean and easy to deal with, we will make the following identifications of $SD$ and $\mathbb{I}_n(P_1)$ with $C([-1,1]^2, M_n(P_0) \oplus M_n(B))$. The identifications are based on the endpoint criteria of each of the $C^*$-algebras. Observe that each $C^*$-algebra has an inner and outer continuous variable and both operate in relation to $P_0$ and $B$, though sometimes not in the same matrix algebra. We identify $SD$ with $C([-1,1]^2, M_n(P_0) \oplus M_n(B))$ by diagonalizing the $P_0$ part as to have the correct size. So $SD$ consist of the following functions $(a(s,t), f(s,t)) \in C([-1,1]^2, M_n(P_0) \oplus M_n(B))$, $a(s,t) \in M_n(P_0)$ and $f(s,t) \in M_n(B)$ for all $(s,t) \in [-1,1]^2$, that follow the constraints

- $a(s,t) = a(s,1) \in 1_{M_n} \otimes P_0$
- $a(-1,t) = 0 = a(1,t)$
- $f(-1,t) = f(1,t) = f(s,-1) = 0$
- $f(s,1) = \psi^{(n)}(a(s,1))$

for all $s, t \in [-1,1]$. In the same way, $\mathbb{I}_n(P_1)$ has the identification $(a(s,t), f(s,t)) \in C([-1,1]^2, M_n(P_0) \oplus M_n(B))$ subject to

- $a(s,t) = a(s,1)$
- $a(-1,t) = 0$
- $a(1,t) \in 1_{M_n} \otimes P_0$
- $f(-1,t) = f(s,-1) = 0$
- $f(1,t) \in 1_{M_n} \otimes B$
- $f(s,1) = \psi^{(n)}(a(s,1))$

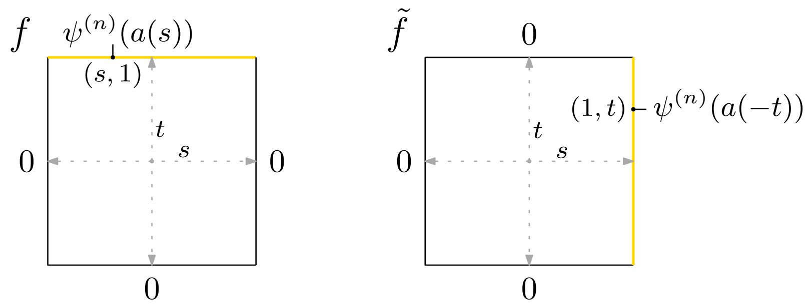
for all $s, t \in [-1,1]$. We write $a(s) := a(s,t)$ for short since the second variable is constant.

*What is the composition $(\mu_n \otimes \mathrm{id}_{P_1}) \circ S\xi$ under this identification?*

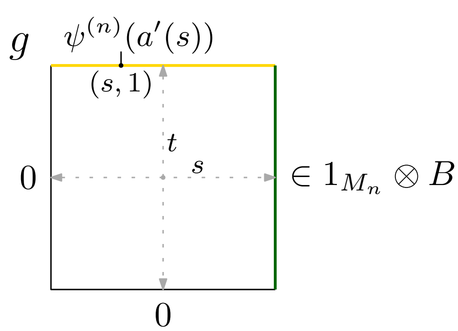
If we drop the identification for a moment and just look at $(a(s), f(s,t)) \in SD$ where $s$ is the suspension argument and $t$ is the $\mathbb{I}_n(B)$ argument coming from $P_1$. Then we see that $(a(s), f(s,t)) \overset{S\xi}{\mapsto} (1_{M_n} \otimes a(s), f(s,t)) \in SM_n(P_1)$. Now $\mu_n \otimes \mathrm{id}_{P_1}$ is just the inclusion, and since our identification above identifies $a(s) \in P_0$ with $1_{M_n} \otimes a(s) \in M_n(P_0)$ we just have the identity. So under our identification in $C([-1,1]^2, M_n(P_0) \oplus M_n(B))$ we just have $(a, f) \mapsto (a, f)$ by $(\mu_n \otimes \mathrm{id}_{P_1}) \circ S\xi$.

*What is the composition* $(\mathrm{id}_{\mathbb{I}_n} \otimes \phi) \circ \overline{S}\eta$ *under this identification?*
See that $\overline{S}\eta(a, f) = f' \in S\mathbb{I}_n(B)$ where $f'(s,t) = f(-s,t)$ and the isomorphism $i : S\mathbb{I}_n(B) \xrightarrow{\cong} \mathbb{I}_n(SB)$ sends $f \mapsto ((s,t) \mapsto f(t,s))$ as previously discussed. So the composition is $i \circ \overline{S}\eta(a,f) = i(f') = ((s,t) \mapsto f'(t,s) = f(-t,s))$. Furthermore, $\mathrm{id}_{\mathbb{I}_n} \otimes \phi(f(t,-s))$ is the inclusion $SB \hookrightarrow CB$ entrywise into $P_1$. So in the identification we have that $(\mathrm{id}_{\mathbb{I}_n} \otimes \phi) \circ i \circ \overline{S}\eta$ maps $(a,f) \mapsto (0, \tilde{f})$ where $\tilde{f}(s,t) = f(-t,s)$.

Working in our identification, let $(a,f) \in (\mu_n \otimes \mathrm{id}_{P_1}) \circ S\xi(SD) \subseteq \mathbb{I}_n(P_1)$ and $(0,\tilde{f}) \in (\mathrm{id}_{\mathbb{I}_n} \otimes \phi) \circ i \circ \overline{S}\eta(SD) \subseteq \mathbb{I}_n(P_1)$. See that $f$ and $\tilde{f}$ behaves almost identically on the boundary of $[-1,1]^2$




but with a 90 degree rotation. When defining the homotopy, we need to make sure that it is in our identification in $\mathbb{I}_n(P_1)$. That is, we need to make sure that $(a', g) \in \mathbb{I}_n(P_1)$ has the boundary conditions




and also the appropriate conditions on $a'$. The idea is to define a homotopy of $f$ onto $\tilde{f}$ and then define a homotopy taking $a$ to 0 whist being compatible with the above.
Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the the rotation function given by multiplication with

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

This rotates vectors by angle $\theta$ in the counterclockwise direction. Since we want to rotate $[-1,1]^2$ onto itself, we can not directly use $R_\theta$ since it may give vectors outside of $[-1,1]^2$. $[-1,1]^2$ is the unit ball in the max norm $\|\cdot\|_\infty$ on $\mathbb{R}^2$ and thus we want vectors to retain their max norm after rotation. This is the idea behind defining $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ for $\theta \in [0, \pi/2]$ in the following way

$$H_\theta(s,t) = \frac{||(s,t)||_\infty}{||R_\theta(s,t)||_\infty} R_\theta(s,t)$$

for $(s,t) \neq (0,0)$ in $[-1,1]^2$ and setting $H_\theta(0,0) = (0,0)$. We see that $H_\theta([-1,1]^2) = [-1,1]^2$ and importantly for our purposes, we have that $H_0 = \mathrm{id}_{\mathbb{R}^2}$ and $H_{\pi/2}(s,t) = (-t,s)$ which can be seen since $R_{\pi/2}(s,t) = (-t,s)$. This will rotate the $f$ functions in our homotopy.

For the $P_0$ part encoded by the continuous function $a$, we will define $F_\theta : [-1,1] \to [-1,1]$ for $\theta \in [0,\pi/2]$. Compatibility $f(s,1) = \psi^{(n)}(a(s,1))$ imposes the condition that $H_\theta(s,1) = (F_\theta(s),1)$. Writing out

$$H_\theta(s,1) = \frac{1}{\max(|s\,\cos(\theta) - \sin(\theta)|, |s\sin(\theta) + \cos(\theta)|)} \begin{bmatrix} s\cos(\theta) - \sin(\theta) \\ s\sin(\theta) + \cos(\theta) \end{bmatrix}$$

we see that if $H_\theta(s,1) = (*,1)$, then equivalently we have $s\sin(\theta) + \cos(\theta) \geq |s\,\cos(\theta) - \sin(\theta)|$ and in this case we get

$$H_\theta(s,1) = \begin{bmatrix} \frac{s\cos(\theta)-\sin(\theta)}{|s\sin(\theta)+\cos(\theta)|} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s\cos(\theta)-\sin(\theta)}{s\sin(\theta)+\cos(\theta)} \\ 1 \end{bmatrix}$$

The inequality can be solved giving the range $r_0 \leq s \leq 1$ where $r_0 = \frac{\sin(\theta)-\cos(\theta)}{\sin(\theta)+\cos(\theta)}$. So we define $F_\theta : [-1,1] \to [-1,1]$ by setting $F_\theta(s) = -1$ for $-1 \leq s \leq r_0$ and setting $F_\theta(s) = \frac{s\cos(\theta)-\sin(\theta)}{s\sin(\theta)+\cos(\theta)}$ for $r_0 \leq s \leq 1$. It can be seen that $F_\theta([-1,1]) = [-1,-r_\theta]$ while $r_0 = -1$ and $r_{\pi/2} = 1$, so we have $F_0 = \mathrm{id}_{[-1,1]}$ and $F_{\pi/2}$ is constant on $-1$. If $s \in [-1,1]$ such that $H_\theta(s,1) = (*,1)$ then we have $f(H_\theta(s,1)) = f(F_\theta(s),1) = \psi^{(n)}(a(F_\theta(s)))$.

Define the homotopy $\Phi_\theta : SD \to \mathbb{I}_n(P_1)$ for $\theta \in [0,\pi/2]$ by $\Phi_\theta(a,f) = (a_\theta, f_\theta)$ where

$$a_\theta(s) = a(F_\theta(s)) \quad \text{and} \quad f_\theta(s,t) = f(H_\theta(s,t))$$

We will check that the homotopy is well defined. The map is easily seen to be a $*$-homomorphism for all $\theta \in [0,\pi/2]$ since $\Phi_\theta((a,f)+\lambda(a',f')) = \Phi_\theta(a,f) + \lambda\Phi(a',f')$, $\Phi_\theta((a,f)(a',f')) = \Phi_\theta(aa', ff')$ and $\Phi_\theta((a,f)^*) = \Phi_\theta(a,f)^*$.

Now we need to check that the image $\Phi_\theta(SD)$ is a subset of $\mathbb{I}_n(P_1)$ for all $\theta \in [0,\pi/2]$. Let $\theta \in [0,\pi/2]$ and $(a,f) \in SD$, then we have

$$\Phi_\theta(a,f) = (a(F_\theta(s)), f(H_\theta(s,t)))$$

and these agree on the conditions in the identification to be in $\mathbb{I}_n(P_1)$ since

- $a_\theta(-1) = a(F_\theta(-1)) = a(-1) = 0$
- $a_\theta(1) = a(F_\theta(1)) \in 1_{M_n} \otimes P_0$ since $a(s) \in 1_{M_n} \otimes P_0$.
- $f_\theta(-1,t) = f(H_\theta(-1,t)) = 0$ (is immediate looking at the illustration of the boundary of $f$)
- $f_\theta(s,-1) = f(H_\theta(s,-1)) = 0$ (is immediate looking at the illustration of the boundary of $f$)
- $f_\theta(1,t) = f(H_\theta(1,t)) = \psi^{(n)}(a(x)) \in 1_{M_n} \otimes B$ for some $x \in [-1,1]$ and since $a(s) \in 1_M \otimes P_0$ for all $s \in [-1,1]$.
- $f_\theta(s,1) = f(H_\theta(s,1)) = f(F_\theta(s),1) = \psi^{(n)}(a(F_\theta(s))) = \psi^{(n)}(a_\theta(s))$

This proves that the desired diagram commutes and gives us an explicit way to calculate $\phi_i^{(n)} : K_{1-i}(B;\mathbb{Z}/n) \to K_i(P_1;\mathbb{Z}/n)$. □

We are now ready to prove the big theorem and conclusion to the thesis.

**Theorem 6.9.** *If a $C^*$-algebra $A$ satisfies the equivalent conditions of the minimal/-maximal Künneth Theorem (Theorem 4.5), then for any other $C^*$-algebra $B$, we have that $\theta^{A,B}_{\min}/\theta^{A,B}_{\max}$ is an isomorphism and its inverse is $\beta^{A,B}_{\min}/\beta^{A,B}_{\max}$ as featured in the Künneth Theorem.*

By inverse, we mean as a map that sends $x \in K_*(A \otimes_\gamma B)$ to its equivalence class in $\operatorname{coker}(\alpha_\gamma^{\mathrm{A,B}})$ and then by $\theta_\gamma^{A,B}$ to elements in the $\operatorname{Tor}(K_*(A), K_*(B))$.

*Proof.* We will use $\otimes$ instead of $\otimes_{\min}$ or $\otimes_{\max}$ in the proof since the arguments are exactly the same for both the minimal and maximal versions of the theorem. So let $A$ satisfy either the minimal or maximal version of Theorem 4.5 depending on what version of the proof is desired.

To prove the theorem, we only have to prove it in the $K_0(A \otimes B)$ case, that is, we need to prove that

$$\theta : \substack{\operatorname{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \operatorname{Tor}(K_1(A),K_0(B))} \to \operatorname{coker}\left(\alpha : \substack{K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B)} \to K_0(A \otimes B)\right)$$

as defined in Theorem 6.9, is the inverse of the map

$$\operatorname{coker}\left(\alpha : \substack{K_0(A)\otimes K_0(B) \\ \oplus \\ K_1(A)\otimes K_1(B)} \to K_0(A \otimes B)\right) \to \substack{\operatorname{Tor}(K_0(A),K_1(B)) \\ \oplus \\ \operatorname{Tor}(K_1(A),K_0(B))} \qquad (\star)$$

which takes $a + \operatorname{im}(\alpha) \in \operatorname{coker}(\alpha)$ and sends it to $(\beta^{A,B})_0(a)$ where $(\beta^{A,B})_0$ is the map from Theorem 4.5 statement *3.* restricted to $K_0(A \otimes B)$.

As we have seen previously in the proof of Theorem 4.5, Lemma 4.2 gives us a $*$-homomorphism $\psi : P_0 \to S^2B \otimes \mathcal{K}$ such that $K_*(\psi)$ is surjective. Let $P_1$ be the mapping cone of $\psi$, then we have a short exact sequence

$$0 \longrightarrow S(S^2B \otimes \mathcal{K}) \xrightarrow{\phi} P_1 \xrightarrow{\rho} P_0 \longrightarrow 0$$

giving us the free resolution

$$0 \longrightarrow K_*(P_1) \xrightarrow{K_*(\rho)} K_*(P_0) \xrightarrow{K_*(\psi)} K_*(B) \longrightarrow 0$$

Since $K$-theory ignores double suspensions and tensorising with the compact operators, we will replace $S^2B \otimes \mathcal{K}$ with $B$ in the rest of the proof.
The construction of $\beta^{A,B}$ in Theorem 4.5 starts off with the short exact sequence

$$0 \longrightarrow A \otimes SB \xrightarrow{\mathrm{id}_A \otimes \phi} A \otimes P_1 \xrightarrow{\mathrm{id}_A \otimes \rho} A \otimes P_0 \longrightarrow 0$$

and applying $K$-theory we get

$$K_0(A \otimes P_0) \longrightarrow K_0(A \otimes B) \xrightarrow{(\mathrm{id}_A \otimes \phi)^1_*} K_1(A \otimes P_1) \xrightarrow{(\mathrm{id}_A \otimes \rho)^1_*} K_1(A \otimes P_0)$$

Now see that as in the proof of Theorem 4.5, the image of $K_0(A \otimes P_0) \to K_0(A \otimes B)$ is exactly the image of $\alpha : K_0(A) \otimes K_0(B) \oplus K_1(A) \otimes K_1(B) \to K_0(A \otimes B)$. So the map $(\mathrm{id}_A \otimes \phi)^1_*$ induces and isomorphism

$$\mathrm{coker}(\alpha) \cong \mathrm{im}((\mathrm{id}_A \otimes \phi)^1_*) \cong \ker((\mathrm{id}_A \otimes \rho)^1_*)$$

From the *2.* equivalent statement of Theorem 4.5 we know that $K_1(A \otimes P_j) \cong K_0(A) \otimes K_1(P_j) \oplus K_1(A) \otimes K_0(P_j)$ for $j \in \mathbb{Z}/2$ given by $\alpha$. As in the construction of the map $\beta^{A,B}$ in Theorem 4.5, the free resolution of $K_*(B)$ then gives us an isomorphism $\mathrm{coker}(\alpha) \cong \mathrm{Tor}(K_0(A), K_1(B)) \oplus \mathrm{Tor}(K_1(A), K_0(B))$ which proves that $(\star)$ is an isomorphism since we have followed the same construction of $\beta^{A,B}$ as in Theorem 4.5.
To finish the proof, it is enough to show that the isomorphism $(\star)$ and $\theta$ agree. To accomplice this, we want to see that

$$(x, n, y) \in \mathrm{Tor}(K_0(A), K_1(B)) \oplus \mathrm{Tor}(K_1(A), K_0(B))$$

is mapped to the same element in $\mathrm{coker}(\alpha)$ by $(\star)$ and $\theta$. We will start off by looking at what $\theta$ does to $(x, n, y)$. Let $(x, n, y) \in \mathrm{Tor}(K_i(A), K_{1-i}(B))$ for a $i \in \mathbb{Z}/2$ which determines the summand of the torsion product. Let $\tilde{x} \in K_{1-i}(A; \mathbb{Z}/n)$ and $\tilde{y} \in K_i(B; \mathbb{Z}/n)$ such that $\nu^{(n)}_{1-i,A}(\tilde{x}) = x$ and $\nu^{(n)}_{i,B}(\tilde{y}) = y$. Now from the definition we have $\theta(x, n, y) = [\Theta^{(n)}_{1-i,i}(\tilde{x} \otimes \tilde{y})] \in \mathrm{coker}(\alpha)$. By naturality of $\Theta$ we have that the diagram

$$\begin{tikzcd} K_0(A \otimes B) \arrow[rr, "(\mathrm{id}_A \otimes \phi)^1_*"] & & K_1(A \otimes P_1) \\ K_{1-i}(A; \mathbb{Z}/n) \otimes K_i(B; \mathbb{Z}/n) \arrow[u, "\Theta^{(n)A,B}_{1-i,i}"] \arrow[rr, "\mathrm{id} \otimes \phi^{(n)}_{1-i}"] & & K_{1-i}(A; \mathbb{Z}/n) \otimes K_{i-1}(P_1; \mathbb{Z}/n) \arrow[u, "\Theta^{(n)A,P_1}_{1-i,1-i}"'] \end{tikzcd}$$

commutes and thus the image of $[\Theta^{(n)}_{1-i,i}(\tilde{x} \otimes \tilde{y})] \in \mathrm{coker}(\alpha)$ mapped to $\ker((\mathrm{id}_A \otimes \rho)^1_*) \subseteq K_1(A \otimes P_1)$ by the induced map $x + \mathrm{im}(\alpha) \mapsto (\mathrm{id}_A \otimes \phi)^1_*(x)$ is

$$\Theta^{(n)}_{1-i,1-i}(\tilde{x} \otimes \phi^{(n)}_{1-i}(\tilde{y}))$$

Using Lemma 6.8, we see that $\phi^{(n)}_{1-i}$ can be computed as follows. Choose $p \in K_{1-i}(P_0)$ such that $K_{1-i}(\psi)(p) = \nu^{(n)}_{i,B}(\tilde{y}) = y$, then $z \in K_{1-i}(P_1)$ such that $K_{1-i}(\rho)(z) = np$ and then we have $\phi^{(n)}_{1-i}(\tilde{y}) = -\mu^{(n)}_{1-i,P_1}(z)$. Hence we have

$$\begin{aligned} \Theta^{(n)}_{1-i,1-i}(\tilde{x} \otimes \phi^{(n)}_{1-i}(\tilde{y})) &= -\Theta^{(n)}_{1-i,1-i}(\tilde{x} \otimes \mu^{(n)}_{1-i,P_1}(z)) \\ &\overset{\diamond}{=} \alpha^{A,P_1}(\nu^{(n)}_{1-i,A}(\tilde{x}) \otimes z) = \alpha^{A,P_1}(x \otimes z) \end{aligned}$$

where $\diamond$ is given by Lemma 6.4 and so we have

$$x \otimes z \in \ker \begin{pmatrix} \begin{smallmatrix} K_0(A) \otimes K_1(P_1) \\ \oplus \\ K_1(A) \otimes K_0(P_1) \end{smallmatrix} \to \begin{smallmatrix} K_0(A) \otimes K_1(P_0) \\ \oplus \\ K_1(A) \otimes K_0(P_0) \end{smallmatrix} \end{pmatrix}$$

where the map is induced by $(\mathrm{id}_A \otimes \rho)^1_*$ since $\alpha^{A,P_1}$ and $\alpha^{A,P_0}$ are isomorphisms.

Now we want to compute the image of $(x, n, y) \in \mathrm{Tor}(K_i(A), K_{1-i}(B))$ via the isomorphism $(\star)$. First we first need to calculate $\delta(x, n, y)$ in the following diagram

$$\begin{tikzcd} \mathrm{Tor}(K_i(A), K_{1-i}(B)) \arrow[r, "\delta"] & K_i(A) \otimes K_{1-i}(P_1) \arrow[r, dashed, "\mathrm{id} \otimes K_{1-i}(\rho)"] & K_i(A) \otimes K_{1-i}(P_0) \arrow[r, "\mathrm{id} \otimes K_{1-i}(\psi)"] & K_i(A) \otimes K_{1-i}(B) \end{tikzcd}$$

coming from the free resolution of $K_*(B)$. $\delta$ is the connecting homomorphism from Proposition 2.5 with a concrete description right above the theorem. This description prescribes that we first pick a lift $\tilde{p} \in K_{1-i}(P_0)$ of $y \in K_{1-i}(B)$, then $K_{1-i}(\psi)(n\tilde{p}) = 0$ and so we can find $\tilde{z} \in K_{1-i}(P_1)$ such that $K_{1-i}(\rho)(\tilde{z}) = n\tilde{p}$. Now $\delta(x, n, y) = x \otimes \tilde{z} \in K_i(A) \otimes K_{1-i}(P_1)$. $\delta$ is independent of the choice of $\tilde{p}$ and since we can choose $\tilde{p} = p$, then by extension we can have $\tilde{z} = z$. This gives us

$$\delta(x, n, y) = x \otimes z \in K_i(A) \otimes K_{1-i}(P_1)$$

Now to the important conclusion, observe that $\alpha^{A,P_1}(x \otimes z) \in K_1(A \otimes P_1)$ mapped by the isomorphism $\ker((\mathrm{id}_A \otimes \rho)^1_*) \cong \mathrm{coker}(\alpha)$ from before is exactly $\theta(x, n, y)$. This proves that $\theta$ is the inverse of the isomorphism $(\star)$ and we are done. $\square$

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