DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE

UNIVERSITY OF SOUTHERN DENMARK

MASTER THESIS IN MATHEMATICS

Automorphisms on classifiable C^*-algebras

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June 01, 2024

Resume

Denne specialeafhandling vil fokusere på invarianten $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ for C^*-algebraer introduceret i Classifying *-homomorphisms I: Unital simple nuclear C^*-algebras, af Carrión, Gabe, Schafhouser, Tikuisis, og White. Vi starter med at introducere den nødvendige teori for at forstå og arbejde med $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$. Derefter studere vi invarianten mere abstrakt ved brug af kategoriteori. Vi slutter afhandlingen af med, at kigge på automorfierne på $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ og viser at automorfigruppen $\mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}})$ kun afhænger af KT_u.

Contents

1 Introduction

Given a C^*-algebra, A, with enough regularity conditions, and a discrete group, G, it is an open problem whether a group action on $KT_u(A)$ lifts to a group action on A. The thesis will explore this problem for classifiable C^*-algebras. We show one can always lift group actions to $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A)$, further we will provide an explicit formula for $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A)$ in terms of $\mathrm{KT_u}(A)$. To accomplish this we introduce the necessary framework, mainly focusing on the article Classifying *-homomorphisms I: Unital simple nuclear C^*-algebras by Carrión, Gabe, Schafhouser, Tikuisis, and White, corresponding to sections 2 through 6 . Afterwards we will view some of their constructions in a more abstract setting, leading to our main results, sections 7 and 8.

The thesis is written for people who have basic knowledge of C^*-algebras, roughly corresponding to an introductory course, as well as knowledge of K-theory for C^*-algebras corresponding to the book [RLL00]. The thesis is written in conjunction with a project about total K-theory for C^*-algebras, so to maximise reading enjoyment it is preferable to have knowledge in this area as well. If one does not know of total K-theory for C^*-algebras then the relevant definitions and results can be found in the preliminaries section, but the proofs have been omitted.

2 Preliminaries

The thesis is made in conjunction with a project concerning total K-theory for C^*-algebras and therefore this theory will not be covered in detail. Instead we will summarise the main results and definitions from the project, and the results will have their proves omitted. For the rest of this section A is a C^*-algebra, $n \in \mathbb{N}$ with $n \geq 2$, and $i \in \{0,1\}$. Before one can understand total K-theory we first need to understand K-theory with coefficients.

Definition 2.1 ([CGS⁺23], (2.26))**.** Let $n \in \mathbb{N}$, $n \geq 2$, $i \in \{0,1\}$, and A be a C^*-algebra. Define *K-theory with coefficients*, $K_i(A,\mathbb{Z}_n)$, by

$$K_i(A;\mathbb{Z}_n) := K_{1-i}(\mathbb{I}_n(A)),$$

where $\mathbb{I}_n(A)$ is the dimension drop algebra,

$$\mathbb{I}_n(A) = \{f \in C([0,1], M_n(A)) \mid f(0) \in A \otimes 1_{M_n}, f(1) = 0\}.$$

As with K-theory, we can induce 6-term exact sequences in K-theory with coefficients.

Proposition 2.2 ([CGS⁺23], (2.29))**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $i \in \{0,1\}$, and A be a C^*-algebra, then there exists maps $\mu^{(n)}_{i,A}, \nu^{(n)}_{i,A}$ such that the 6-term sequence*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mu^{(n)}_{0,A}} & K_0(A;\mathbb{Z}_n) & \xrightarrow{\nu^{(n)}_{0,A}} & K_1(A) \\ {\scriptstyle\times n}\uparrow & & & & \downarrow{\scriptstyle\times n} \\ K_0(A) & \xleftarrow[\nu^{(n)}_{1,A}]{} & K_1(A;\mathbb{Z}_n) & \xleftarrow[\mu^{(n)}_{1,A}]{} & K_1(A), \end{array}$$

is exact and natural in A. We call the maps $\mu^{(n)}_{i,A}, \nu^{(n)}_{i,A}$ Bockstein operations.

Before we can define total K-theory we need two more Bockstein operations.

Proposition 2.3 ([CGS⁺23], (2.30))**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $i \in \{0,1\}$, and A be a C^*-algebra. The canonical inclusions $\mathbb{I}_n \cong \mathbb{I}_n \otimes 1_{M_n} \hookrightarrow \mathbb{I}_{mn}$, and $\mathbb{I}_{nm} \hookrightarrow \mathbb{I}_n \otimes M_m$ induces maps*

$$\kappa_{i,A}^{(nm,n)} : K_i(A, \mathbb{Z}_n) \to K_i(A, \mathbb{Z}_{nm}),$$
$$\kappa_{i,A}^{(n,nm)} : K_i(A, \mathbb{Z}_{nm}) \to K_i(A, \mathbb{Z}_n).$$

As in Proposition 2.3 we also call $\kappa_{i,A}^{nm,n}, \kappa_{i,A}^{n,nm}$ Bockstein operations.

We are now able to define the total K-theory of a C^*-algebra A.

Definition 2.4 ([CGS⁺23], (Def 2.14))**.** The *total K-theory*, $\underline{K}(A)$, of a C^*-algebra, A, is the collection of abelian groups $K_0(A), K_1(A)$, and $K_i(A;\mathbb{Z}_n)$ for $i \in \{0,1\}$ and $n \geq 2$ together with the Bockstein operations $\mu_{i,A}^{(n)}, \nu_{i,A}^{(n)}, \kappa_{i,A}^{(nm,n)}, \kappa_{i,A}^{(n,nm)}$ for $m \geq 2$. The total K-theory of A is the collection

$$\underline{K}(A) := \left(K_0(A), K_1(A), \left(K_i(A;\mathbb{Z}_n), \mu_{i,A}^{(n)}, \nu_{i,A}^{(n)}, \kappa_{i,A}^{(nm,n)}, \kappa_{i,A}^{(n,nm)} \right)_{\substack{i\in\{0,1\}\\ n,m\geq 2}} \right).$$

A Λ-*morphism*, $\underline{\alpha} : \underline{K}(A) \to \underline{K}(B)$ is a collection of group homomorphisms $\alpha_i : K_i(A) \to K_i(B)$ and $\alpha_i^{(n)} : K_i(A;\mathbb{Z}_n) \to K_i(B;\mathbb{Z}_n)$ for $i \in \{0,1\}$ and $n \geq 2$ such that the diagrams

$$\begin{array}{ccc} K_i(A) & \xrightarrow{\alpha_i} & K_i(B) \\ \downarrow{\scriptstyle \mu_{i,A}^{(n)}} & & \downarrow{\scriptstyle \mu_{i,B}^{(n)}} \\ K_i(A;\mathbb{Z}_n) & \xrightarrow{\alpha_i^{(n)}} & K_i(B;\mathbb{Z}_n), \end{array} \qquad \begin{array}{ccc} K_i(A;\mathbb{Z}_n) & \xrightarrow{\alpha_i^{(n)}} & K_{1-i}(B;\mathbb{Z}_n) \\ \downarrow{\scriptstyle \nu_{i,A}^{(n)}} & & \downarrow{\scriptstyle \nu_{i,B}^{(n)}} \\ K_i(A) & \xrightarrow{\alpha_{1-i}} & K_{1-i}(B), \end{array}$$

$$\begin{array}{ccc} K_i(A;\mathbb{Z}_n) & \xrightarrow{\alpha_i^{(n)}} & K_i(B;\mathbb{Z}_n) \\ \downarrow{\scriptstyle \kappa_{i,A}^{(nm,n)}} & & \downarrow{\scriptstyle \kappa_{i,B}^{(nm,n)}} \\ K_i(A;\mathbb{Z}_{nm}) & \xrightarrow{\alpha_i^{(nm)}} & K_i(B;\mathbb{Z}_{nm}), \end{array} \qquad \begin{array}{ccc} K_i(A;\mathbb{Z}_{nm}) & \xrightarrow{\alpha_i^{(nm)}} & K_i(B;\mathbb{Z}_{nm}) \\ \downarrow{\scriptstyle \kappa_{i,A}^{(n,nm)}} & & \downarrow{\scriptstyle \kappa_{i,B}^{(n,nm)}} \\ K_i(A;\mathbb{Z}_n) & \xrightarrow{\alpha_i^{(n)}} & K_i(B;\mathbb{Z}_n), \end{array}$$

commute for $m \geq 2$.

Having defined total K-theory let's see some results.

Proposition 2.5 ([CGS⁺23], (2.35))**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $i \in \{0,1\}$ and A be a C^*-algebra. Then there exists maps $\overline{\mu}_{i,A}^{(n)} : K_i(A)/nK_i(A) \to K_i(A;\mathbb{Z}_n)$ and $\overline{\nu}_{i,A}^{(n)} : K_i(A;\mathbb{Z}_n) \to \operatorname{Tor}(K_{1-i}(A), \mathbb{Z}_n)$ such that the sequence*

$$0 \longrightarrow K_i(A) \otimes \mathbb{Z}_n \xrightarrow{\overline{\mu}_{i,A}^{(n)}} K_i(A;\mathbb{Z}_n) \xrightarrow{\overline{\nu}_{i,A}^{(n)}} \operatorname{Tor}(K_{1-i}(A), \mathbb{Z}_n) \longrightarrow 0.$$

is short exact moreover $\mu_{i,A}^{(n)} = \overline{\mu}_{i,A}^{(n)} \circ \pi$ and $\nu_{i,A}^{(n)} = \iota_{1-i,A}^{(n)} \circ \overline{\nu}_{i,A}^{(n)}$, where $\pi : K_i(A) \to K_i(A) \otimes \mathbb{Z}_n$ is the projection and $\iota_{1-i,A}^{(n)} : \operatorname{Tor}(K_{1-i}(A), \mathbb{Z}_n) \to K_{1-i}(A)$ is the inclusion.

The next result is very powerful, and will be paramount when proving some of our big results in the later sections.

Proposition 2.6 ([CGS⁺23], Proposition 2.15)**.** *Let A, B be C^*-algebras and $i \in \{0,1\}$. Then any homomorphisms $\alpha_i : K_i(A) \to K_i(B)$ can be extended to a Λ-morphism $\underline{\alpha} : \underline{K}(A) \to \underline{K}(B)$. This is an isomorphism when α is.*

A big part of this result is the fact that isomorphisms in K-theory lift to Λ-isomorphisms which we will use when working with automorphism groups. A possibly innocuous, but quite useful, lemma is the following.

Lemma 2.7 (Splitting lemma)**.** *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\psi_1} & B_1 & \underset{s_1}{\overset{\varphi_1}{\rightleftarrows}} & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{\psi_2} & B_2 & \underset{s_2}{\overset{\varphi_2}{\rightleftarrows}} & C_2 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of abelian groups with exact rows and s_i be right splits for $i \in \{1, 2\}$. Then there exists homomorphisms $\sigma_i : B_i \to A_i$ such that

$$\psi_i \circ \sigma_i = \mathrm{id}_{B_i} - s_i\varphi_i.$$

Further σ_i is a left split and, $\alpha\sigma_1 = \sigma_2\beta$.

With the preliminaries taking care of we are ready to move into the main part of the thesis.

3 Aff$T(A)$

One of the constituents of the total invariant is all real valued affine continuous functions of the trace space, which we will devote this section to understand. First we need to understand order unit spaces and their relation to affine functions on state spaces.

3.1 Order unit spaces

The goal of this subsection is to introduce order unit spaces and ultimately show Kadison duality, which relates an order unit space and affine functions on the state space. Most results comes from the book: Compact convex sets and boundary integrals by Erik M. Alfsen, [Alf71], from the first part of chapter 2.

Before we can see what an order unit space is, we first need to define partially ordered vector spaces.

Definition 3.1. Let A be a vector space over $\mathbb{R}$ and let $\leq$ be a partial order on A. We say that A is a *partially ordered vector space* if for all $a, b, c \in A$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$ we have

$$a \leq b \implies a + c \leq b + c,$$
$$a \leq b \implies \lambda a \leq \lambda b.$$

If $0 \leq a$ then we say a is *positive* and we denote by A^+ the collection of positive elements of A, further if $a \leq 0$ we say that a is *negative* and we denote the collection of negative elements by A^-.

We are now able to define an order unit.

Definition 3.2 ([Alf71])**.** Let A be a partially ordered vector space over $\mathbb{R}$ and let $I \subseteq A$ be a subspace. We say that I is an *order ideal* if for $a, b \in I$ and $c \in A$ we have that

$$a \leq c \leq b \implies c \in I.$$

Denote by $I(a)$ the order ideal generated by $a \in A$. If there exists $e \in A^+$ such that $I(e) = A$ we say e is an *order unit.*

Contrary to what the name order unit space might suggest, it is not just a partially ordered vector space with an order unit, we need one more adjective.

Definition 3.3 ([Alf71])**.** Let A be a partially ordered vector space with order unit e. We say that A is *Archimedean*[1] if for all $n \in \mathbb{N}$

$$na \leq e \implies a \in A^-.$$

Having seen the above we can finally define the name of the subsection, an order unit space.

Definition 3.4 ([Alf71])**.** Let A be a partially ordered vector space with order unit e. If A is Archimedean we say that A is an *order unit space* and we will denote it by (A, e).

A nice property of order unit spaces is that they admit a norm.

Proposition 3.5 ([Alf71](II.1.2))**.** *Let (A, e) be an order unit space. This space admits a norm, called the order unit norm, given by*

$$||a|| = \inf\{\lambda > 0 \mid -\lambda e \leq a \leq \lambda e\}$$

which satisfies the relation

$$-||a||e \leq a \leq ||a||e.$$

Proof. We start by defining two maps $l, m : A \to \mathbb{R}$ given by

$$l(a) = \sup\{\alpha \in \mathbb{R} \mid \alpha e \leq a\} \quad m(a) = \inf\{\beta \in \mathbb{R} \mid a \leq \beta e\}.$$

By definition of our proposed norm we get that $||a|| = \max(m(a), -l(a))$, and hence $||a||$ is a semi norm. We show the semi norm satisfies $-||a||e \leq a \leq ||a||e$ as this will imply that $|| \cdot ||$ is a norm. Let $n \in \mathbb{N}$

$$a \leq m(a)e + \frac{1}{n}e,$$

thus

$$na \leq nm(a)e + e,$$

hence

$$n(a - m(a)e) \leq e.$$

Since A is Archimedean $a \leq m(a)e$, an analogous computation shows $l(a)e \leq a$ hence

$$-||a||e \leq l(a)e \leq a \leq m(a)e \leq ||a||e.$$

□

As mentioned above a major player in Kadison duality is the real valued continuous affine functions.

Definition 3.6 ([Alf71])**.** Let X be a locally convex Hausdorff space over $\mathbb{R}$. Let $K, K' \subseteq X$ be convex subsets of X such that $K \subseteq K'$. Denote by $\text{Aff}(K, K')$ the vector space of all restrictions to K of real valued continuous affine functions on K'. We will write $\text{Aff}(K)$ or $\text{Aff}K$ to denote $\text{Aff}(K, K)$ for simplicity.

[1]One can also define Archimedean for partially ordered vector spaces without order units, as follows: A is Archimedean if the negative elements, $a \in A^-$, are the only ones for which $\{\lambda a \mid \lambda \in \mathbb{R}^+\}$ has an upper bound. One can show that for order unit spaces being Archimedean in the sense just described is equivalent to our definition. As we will work with order unit spaces we choose this as our definition.

A theme in the thesis is that when considering new objects, in this case order unit spaces, we will also consider their structure preserving maps, and in most cases these will be of more interest.

Definition 3.7 ([Alf71]). Let $(A, e), (A', e')$ be order unit spaces, we say that a linear map $\varphi : (A, e) \to (A', e')$ is *positive* or *order preserving*, if $a \le a' \implies \varphi(a) \le \varphi(a')$ and we will denote it by $\varphi \ge 0$. A if φ is bijective we call φ an order isomorphism if $\varphi, \varphi^{-1} \ge 0$.

Let's see some results concerning positive maps.

Proposition 3.8 ([Alf71](II.1.3)). *Let $(A, e), (A', e')$ be order unit spaces and let $\varphi : (A, e) \to (A', e')$ be a linear map with $\varphi(e) = e'$. Then $\varphi \ge 0$ if and only if $||\varphi|| = 1$.*

Proof. We start by assuming φ is positive. Let $a \in A$ such that $||a|| \le 1$, then $-e \le a \le e$ and since φ is positive, unital $-e' \le \varphi(a) \le e'$ hence $||\varphi(a)|| \le 1$ and thus $||\varphi|| \le 1$. Using that φ is unital we get $||\varphi(e)|| = ||e'|| = 1$ thus $||\varphi|| \ge 1$ hence $||\varphi|| = 1$.

Assume $||\varphi|| = 1$. Let $a \in A^+$ be a positive element. We can without loss of generality assume $||a|| \le 1$ indeed if $||a|| > 1$ we normalise a to $\widetilde{a} = \frac{a}{||a||}$ and as φ is linear we can work with $\varphi(\widetilde{a})$ instead. As $\|a\| \le 1$ then $0 \le e - a \le e$ and thus $||e - a|| \le 1$. Then $||\varphi(e - a)|| \le 1$ since $\|\varphi\| = 1$ thus $\varphi(e - a) \le e'$ and since φ is unital and linear $0 \le \varphi(a)$ hence φ is positive. ☐

We also have a result about order isomorphisms.

Corollary 3.9 ([Alf71](II.1.4)). *Let $(A, e), (A', e')$ be order unit spaces and let $\varphi : (A, e) \to (A', e')$ be a bijective unital linear map. Then φ is an order isomorphism if and only if φ is an isometry.*

Proof. Assume φ is an isometry. Then $||\varphi|| = ||\varphi^{-1}|| = 1$ by Proposition 3.8 both are positive maps.

Assume both φ and φ^{-1} are positive. Let $\|a\| = \alpha$ as φ is positive unital thus, $-\alpha e' \le \varphi(a) \le \alpha e'$. Assume for contradiction $\|\varphi(a)\| \ne \alpha$. Then there exists some $\alpha' \in \mathbb{R}$ such that $0 \le \alpha' < \alpha$ and $-\alpha' e' \le \varphi(a) \le \alpha' e'$. Since φ^{-1} is positive unital $-\alpha' e \le a \le \alpha' e$ which implies $\|a\| \le \alpha'$, contradiction $\|a\| = \alpha$. Thus $\|a\| = \|\varphi(a)\|$. ☐

A particularly nice subset of positive maps are states.

Definition 3.10 ([Alf71]). Let (A, e) be an order unit space. A positive functional $p : A \to \mathbb{R}$ with $p(e) = 1$ is called a *state* on (A, e). The collection of states on (A, e) is called the *state space* and will be denoted by $S(A, e)$ or just S when there is no ambiguity. Note $S(A, e)$ form a w^*-compact convex subset of the dual space, A^*.

Remark: Proposition 3.8 gives us the norm of a state is 1.

Kadison duality connects order unit spaces with affine functions over their state space. Hence we need a bridge between these spaces.

Definition 3.11 ([Alf71]). Let (A, e) be an order unit space and X be a compact Hausdorff space. Let $\rho : (A, e) \to (C_{\mathbb{R}}(X), 1_X)$ be a linear map such that $\rho(e) = 1_X$ and ρ is an isometry. Note that this implies that ρ is an order isomorphism on it's image. We call such a pair (ρ, X) a *functional representation* of (A, e).

Lets see an example of this.

Example 3.12. Let (A, e) be an order unit space. Define $\rho : (A, e) \to (S(A, e), \widehat{e})$ by $\rho(a) = \widehat{a}$, where $\widehat{a}(p) = p(a)$ for $p \in S$, that is $\widehat{a}$ is the evaluation of a. We call this functional representation the *canonical representation over the state space* and denote it by (ρ, S).

We have two result left to show before we show Kadison duality, the first has to do with affine functions on a compact space.

Lemma 3.13. *Let X be a compact Hausdorff space, then* $\mathrm{Aff}(X)$ *is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Aff}(X)$ be a Cauchy sequence. We know $\mathrm{Aff}(X) \subset C_{\mathbb{R}}(X)$ where the latter is complete, so $(f_n)_{n\in\mathbb{N}}$ converges to some continuous function, $f \in C_{\mathbb{R}}(X)$. We show f is affine. For each $n \in \mathbb{N}$, $p_1, p_2 \in X$ and $t \in [0, 1]$ we have

$$f_n(tp_1 + (1-t)p_2) = tf_n(p_1) + (1-t)f_n(p_2).$$

Since $f_n(tp_1 + (1-t)p_2) \to f(tp_1 + (1-t)p_2)$ and $tf_n(p_1) + (1-t)f_n(p_2) \to tf(p_1) + (1-t)f(p_2)$ then $f(tp_1 + (1-t)p_2) = tf(p_1) + (1-t)f(p_2)$. □

The next is a technical result for which we omit the proof.

Lemma 3.14 ([Alf71](I.1.5))**.** *Let X be a locally convex Hausdorff space, $K \subseteq X$ be a compact convex set, and $f \in$* $\mathrm{Aff}(K)$*. Then there exists an increasing sequence $(f_n)_{n\in\mathbb{N}} \subset$* $\mathrm{Aff}(K, X)$ *converging uniformly to f. This yields that* $\mathrm{Aff}(K, X)$ *is uniformly dense in* $\mathrm{Aff}(K)$*.*

We are now ready to prove our main result.

Theorem 3.15 ((Kadison duality) [Alf71](II.1.8))**.** *Let (A,e) be an order unit space and let (ρ, S) be the canonical representation over it's state space. Then it's range, $\rho(A)$, consists of all those w^*-continuous affine functions which can be extended to w^*-continuous linear functionals on the dual space. In particular we have $\rho(A) =$* $\mathrm{Aff}(S)$ *if and only if (A, e) is complete in order unit norm.*

Proof. We start by recalling that by definition, the image of ρ is all evaluation functionals restricted to the state space and these are exactly the w^*-continuous linear functionals on A^*, hence we have shown the first part. Before showing the if and only if, we show $\rho(A) = \mathrm{Aff}(S, A^*)$, note $\rho(A) \subseteq \mathrm{Aff}(S, A^*)$. For the converse consider the set

$$H := \{f \in A^* \mid f(e) = 1\},$$

and note that $S \subset H$. Recall that any continuous affine functional $\psi : A^* \to \mathbb{R}$ can be decomposed into a linear functional on the dual space plus some real number, i.e

$$\psi(f) = \Lambda(f) + c$$

for some $c \in \mathbb{R}$ and $\Lambda \in A^{**}$. For each $x \in \mathbb{R}$ define $\Lambda_x : A^* \to \mathbb{R}$ by

$$\Lambda_x(f) = \Lambda(f) + xf(e).$$

Λ_x is linear, indeed

$$\Lambda_x(\alpha f + g) = \Lambda(\alpha f + g) + x((\alpha f + g)(e)) = \alpha(\Lambda(f) + xf(e)) + \Lambda(g) + xg(e) = \alpha\Lambda_x(f) + \Lambda_x(g).$$

Observe $\Lambda_c|_H = \psi|_H$, thus for each w^* continuous affine functional there exists a w^* continuous linear functional such that they agree on H, hence they agree on S, i.e $\mathrm{Aff}(S, A^*) \subseteq \rho(A)$.

Assume A is complete. By Lemma 3.14 $\text{Aff}(S, A^*)$ is uniformly dense in $\text{Aff}(S)$ hence it suffices show that $\text{Aff}(S, A^*)$ is closed. Let $(f_n)_{n\in\mathbb{N}} \subset \text{Aff}(S, A^*)$ be a sequence of functions and assume $f_n \to f$ uniformly. Since $\rho(A) = \text{Aff}(S, A^*)$ we can find $(a_n)_{n\in\mathbb{N}} \subset A$ such that $\rho(a_n) = f_n$. As ρ is an isometry

$$||a_n - a_m|| = ||\rho(a_n) - \rho(a_m)|| \leq ||\rho(a_n) - f|| + ||\rho(a_m) - f||.$$

Since $\rho(a_n)$ converges to f then for any $\varepsilon > 0$ we get $||a_n - a_m|| < \varepsilon$ and hence $(a_n)_{n\in\mathbb{N}}$ is Cauchy. By completeness of A, $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in A$, thus $\rho(a_n) \to \rho(a) = f$, hence $\text{Aff}(S, A^*)$ is uniformly closed.
Assume $\text{Aff}(S, A^*) = \text{Aff}(S)$, we wish to show that A is complete. Let $(a_n)_{n\in\mathbb{N}} \subset A$ be a Cauchy sequence. As ρ is an isometry, $(\rho(a_n))_{n\in\mathbb{N}} \subset \text{Aff}(S, A^*) = \text{Aff}(S)$ is a Cauchy sequence in $\text{Aff}(S)$. As $\text{Aff}(S)$ is complete by Lemma 3.13 $(\rho(a_n))_{n\in\mathbb{N}}$ converges to some $g \in \text{Aff}(S)$, then since $g \in \rho(A)$ we can find an $a \in A$ such that $\rho(a) = g$. For any $\varepsilon > 0$,

$$||a_n - a|| = ||\rho(a_n) - \rho(a)|| < \varepsilon,$$

hence A is complete. □

We are now finished with our tour of abstract order unit spaces, and we will move to consider trace spaces for C^*-algebras.

3.2 Trace Spaces

In this subsection we apply the results of the previous subsection to a specific example of an order unit space, namely $\text{Aff}T(A)$ where $T(A)$ is the trace space for a C^*-algebra A. Our goal for the section is to show the duality between $T(A)$ and Aff $T(A)$ so that we may work with the latter through the thesis. In this section let A be a unital C^*-algebra and let $T(A)$ be the set of tracial states on A. We start by showing that $\text{Aff}T(A)$ forms an order unit space.

Proposition 3.16. *Let A be a unital C^*-algebra and let $T(A)$ denote the space of tracial states on A. Then $\text{Aff}T(A)$ is an order unit space, with the partial order $f \leq g$ if $f(\tau) \leq g(\tau)$ for all $\tau \in T(A)$.*

Proof. We first show $(\text{Aff}T(A), \leq)$ forms an ordered vector space. Let $f, g, h \in \text{Aff}T(A)$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. For $\tau \in T(A)$, $f \leq g$ implies $f(\tau) \leq g(\tau)$, and as $\mathbb{R}$ is an ordered vector space $f(\tau) + h(\tau) \leq g(\tau) + h(\tau)$ which is exactly $f + h \leq g + h$. Again as $\mathbb{R}$ is an ordered vector space $\lambda f(\tau) \leq \lambda g(\tau)$ hence $\lambda f \leq \lambda g$. We show $(\text{Aff}T(A), \leq)$ admits an order unit let $e = \mathbb{1}_{T(A)}$, the constant 1 function. Since $I(e)$ is a vector subspace of $\text{Aff}T(A)$ then all constant functions are in $I(e)$. Let $f \in \text{Aff}T(A)$ and recall $T(A)$ is compact in the w^*-topology, hence

$$\sup_{\tau\in T(A)} \{f(\tau)\} < \infty.$$

Thus there exists some constant functions g, h such that

$$g \leq f \leq h.$$

Thus $I(e) = \text{Aff}T(A)$, hence e is an order unit for $(\text{Aff}T(A), \leq)$.
We show $\text{Aff}T(A)$ is Archimedean. Let $f \in \text{Aff}T(A)$ such that for all $n \in \mathbb{N}$, $nf \leq e$. By definition of $\leq$ for all $\tau \in T(A)$, $nf(\tau) \leq 1$ which implies $f(\tau) \leq 0$. □

The next part of this thesis will be devoted to proving Proposition 2.1 , from [CGS+23], which states that Aff$T(A)$ is evaluation functionals of self-adjoint elements. This proof requires introducing some new notions. So we move away from Alfsens book and over to Gert Pedersens C^*-algebras and their automorphism groups.

Definition 3.17 ([Ped79](3.1.1))**.** Let A be a unital C^*-algebra and let $\varphi \in A^*$ be a bounded linear functional. Define the adjoint φ^* by

$$\varphi^*(a) = \overline{\varphi(a^*)}.$$

We say φ is selfadjoint if $\varphi^* = \varphi$.

An important result needed to prove Proposition 2.1 is how to decompose bounded selfadjoint linear functionals into positive linear functionals, which is known as the Jordan decomposition.

Theorem 3.18 ([Ped79](3.2.5)(Jordan decomposition))**.** *Let A be a unital C^*-algebra, and $\varphi : A \to \mathbb{C}$ a bounded selfadjoint linear functional. Then there exists a unique pair of positive linear functionals φ_+, φ_- such that $\varphi = \varphi_+ - \varphi_-$ and $||\varphi|| = ||\varphi_+|| + ||\varphi_-||$.*

However to prove this theorem we need two lemmas.

Lemma 3.19 ([Ped79](3.2.2))**.** *The unit ball of $(A_{sa})^*$ is the convex span of $S(A)$ and $-S(A)$.*

Proof. Let K denote the convex span of $S(A)$ and $-S(A)$. As states are positive and have norm equal to 1 K is a subset of the unit ball of $(A_{sa})^*$ and as $S(A)$ and $-S(A)$ are weak* compact K also is weak* compact. Let $x \in A_{sa}$, $\lambda \in \sigma(x)\backslash\{0\}$, and $B = C^*(1, x)$ the C^*-algebra generated by 1 and x. Define a *-homomorphism $\omega : B \to \mathbb{C}$ on generators, $\omega(x) = \lambda$ and $\omega(1) = 1$. Then ω is a state on B and by Proposition 3.8 $||\omega|| = 1$. Applying the Hanh-Banach extension theorem, extend ω to a state φ on A, hence

$$\sigma(x)\backslash\{0\} \subset \{\varphi(x) | \varphi \in S(A)\}.$$

As x is selfadjoint the spectral radius formula tells us $||x|| = r(x) = \sup\{|\varphi(x)| \mid \varphi \in S(A)\}$. With the set-up done assume for contradiction there exists a ψ in the unit ball of $(A_{sa})^*$ and not in K. Using the Hanh-Banach separation theorem, separate K and the unit ball of $(A_{sa})^*$. Hence there exists some x in A_{sa} and $\alpha \in \mathbb{R}$ such that $\psi(x) > \alpha$ and $\varphi(x) \leq \alpha$ for all $\varphi \in K$. As K is the convex span of $S(A)$ and $-S(A)$, K is symmetric hence $|\varphi(x)| \leq \alpha$ and by the spectral radius formular $||x|| \leq \alpha$. But then

$$|\psi(x)| \leq \|\psi\|\|x\|| \leq \alpha,$$

contradicting that $\psi(x) > \alpha$. ☐

Now for the next lemma.

Lemma 3.20 ([Ped79](3.2.3))**.** *Let A be a unital C^*-algebra and let φ and ψ be positive functionals on A. Then $||\varphi - \psi|| = ||\varphi|| + ||\psi||$ if and only if for all $\varepsilon > 0$ there exists $z \in A^1_+$ such that $\varphi(1 - z) < \varepsilon$ and $\psi(z) < \varepsilon$.*

Proof. Assume $||\varphi - \psi|| = ||\varphi|| + ||\psi||$ and let $\varepsilon > 0$. As $\varphi - \psi \in (A_{sa})^*$ we can find an x in A^1_{sa} such that

$$||\varphi - \psi|| \leq \varphi(x) - \psi(x) + \varepsilon.$$

Recall as φ and ψ are positive they attain their norm at the identity, hence

$$\varphi(1) + \psi(1) = \|\varphi\| + \|\psi\| = ||\varphi - \psi|| \leq \varphi(x) - \psi(x) + \varepsilon,$$

so $\varphi(1-x) + \psi(1+x) \leq \varepsilon$. By the spectral mapping theorem $1-x$ and $1+x$ are positive elements less than 2. Define $z := \frac{1}{2}(1+x)$ and note $z \in A^1_+$,

$$\varphi(1-z) = \varphi\left(1 - \frac{1}{2}(1+x)\right) = \frac{1}{2}\varphi(1-x) < \varepsilon,$$
$$\psi(z) = \psi\left(\frac{1}{2}(1+x)\right) = \frac{1}{2}\psi(1+x) < \varepsilon.$$

Assume for every $\varepsilon > 0$ there exists $z \in A^1_+$ such that $\varphi(1-z) < \varepsilon$ and $\psi(z) < \varepsilon$. By the triangle inequality it suffices to show $||\varphi - \psi|| \geq ||\varphi|| + ||\psi||$. Using our assumption

$$\begin{aligned} ||\varphi|| + ||\psi|| &= \varphi(1) + \psi(1) \\ &= \varphi(2 - 1 - 2z + 2z) + \psi(1 - 2z + 2z) \\ &= \varphi(2z-1) + 2\varphi(1-z) + \psi(1-2z) + 2\psi(z) \\ &= \varphi(2z-1) + \psi(1-2z) + 4\varepsilon \\ &= (\varphi - \psi)(2z-1) + 4\varepsilon. \end{aligned}$$

By the spectral mapping theorem $\sigma(2z-1) \subseteq [-1,1]$, $\sigma(1-2z) \subseteq [-1,1]$ hence both elements are self-adjoint and spectral radius tells us $||1-2z|| \leq 1$. Thus

$$(\varphi - \psi)(2z-1) + 4\varepsilon \leq ||\varphi - \psi|| + 4\varepsilon.$$

As ε was arbitrary $||\varphi|| + ||\psi|| \leq ||\varphi - \psi||$. □

We are now ready to prove the Jordan decomposition.

Proof of Jordan decomposition ([Ped79]3.18). We assume without loss of generality that the $||\varphi|| = 1$, indeed if the norm of φ is different from 1 define $\widetilde{\varphi}$ as the normalisation of φ. Thus Lemma 3.19 applies and φ can be written as a convex combination of positive functionals in the unit ball

$$\varphi = \alpha\varphi_1 - (1-\alpha)\varphi_2,$$

for $\alpha \in [0,1]$. Letting $\varphi_+ = \alpha\varphi_1$ and $\varphi_- = (1-\alpha)\varphi_2$, yields a decomposition of φ into positive linear functionals. We show $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$. Using the triangle inequality it suffices to show $\|\varphi_+\| + \|\varphi_-\| \leq \|\varphi\|$. By construction of φ_+, φ_-

$$||\varphi_+|| + ||\varphi_-|| = \alpha||\varphi_1|| + (1-\alpha)||\varphi_2|| = \alpha(||\varphi_1|| - ||\varphi_2||) + ||\varphi_2||.$$

We have two cases $||\varphi_1|| > ||\varphi_2||$ or $||\varphi_1|| \leq ||\varphi_2||$, in case 1, as $\alpha \leq 1$ and $\|\varphi_1\| \leq \|\varphi\| = 1$,

$$\alpha(||\varphi_1|| - ||\varphi_2||) + ||\varphi_2|| \leq ||\varphi_1|| \leq ||\varphi||,$$

and in case 2 $\|\varphi_1\| - \|\varphi_2\| \leq 0$ and $\|\varphi_2\| \leq \|\varphi\| = 1$

$$\alpha(||\varphi_1|| - ||\varphi_2||) + ||\varphi_2|| \leq ||\varphi_2|| \leq ||\varphi||.$$

We show uniqueness. Assume there exists $\varphi_1, \varphi_2, \psi_1, \psi_2$ positive linear functionals such that $\varphi_1 - \psi_1 = \varphi_2 - \psi_2$ and $||\varphi_1|| + ||\psi_1|| = ||\varphi_2|| + ||\psi_2||$. By Lemma 3.20 for all $\varepsilon > 0$ there exists some $z \in A_+^1$ such that $\varphi_1(1-z) < \varepsilon$ and $\psi_1(z) < \varepsilon$. This implies

$$\varphi_2(z) \geq \varphi_2(z) - \psi_2(z) = \varphi_1(z) - \psi_1(z) > \varphi_1(1) - 2\varepsilon,$$
$$\psi_2(1-z) \geq \psi_2(1-z) - \varphi_2(1-z) = \psi_1(1-z) - \varphi_1(1-z) > \psi_1(1) - 2\varepsilon.$$

Thus

$$\varphi_2(z) + \psi_2(1-z) > \varphi_1(1) - 2\varepsilon + \psi_1(1) - 2\varepsilon = ||\varphi_1|| + ||\psi_1|| - 4\varepsilon = ||\varphi_2|| + ||\psi_2|| - 4\varepsilon,$$

hence $\varphi_2(1-z) + \psi_2(z) < 4\varepsilon$. By assumption $\varphi_1(x) - \varphi_2(x) = \psi_1(x) - \psi_2(x)$ for all $x \in A$

$$\begin{aligned}\varphi_1(x) - \varphi_2(x) &= \varphi_1(xz) - \varphi_2(xz) + \varphi_1(x(1-z)) - \varphi_2(x(1-z)) \\ &= \psi_1(xz) - \psi_2(xz) + \varphi_1(x(1-z)) - \varphi_2(x(1-z)).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to $\psi_1(xz)$ and recalling that $z \in A_+^1$,

$$|\psi_1(xz)|^2 \leq \psi_1(x^*x)\psi_1(z^2) \leq ||x||^2||\psi_1||\psi_1(z) \leq ||x||^2||\varphi_1 - \psi_1||\psi_1(z) < ||x||^2||\varphi_1 - \psi_1||\varepsilon,$$

where the second last inequality comes from $||\varphi_1 - \psi_1|| = ||\varphi_1|| + ||\psi_1||$. We bound the other three terms similarly

$$|\psi_2(xz)|^2 \leq \psi_2(x^*x)\psi_2(z^2) \leq ||x||^2||\varphi_2 - \psi_2||\psi_2(z) < ||x||^2||\varphi_1 - \psi_1||4\varepsilon,$$
$$|\varphi_1(x(1-z))|^2 \leq \varphi_1(x^*x)\varphi_1((1-z)^2) \leq ||x||^2||\varphi_1 - \psi_1||\varphi_1(1-z) < ||x||^2||\varphi_1 - \psi_1||\varepsilon,$$
$$|\varphi_2(x(1-z))|^2 \leq \varphi_2(x^*x)\varphi_2((1-z)^2) \leq ||x||^2||\varphi_2 - \psi_2||\varphi_2(1-z) < ||x||^2||\varphi_1 - \psi_1||4\varepsilon.$$

Combining our estimates

$$\begin{aligned}|\varphi_1(x) - \varphi_2(x)| &\leq |\psi_1(xz)| + |\psi_2(xz)| + |\varphi_1(x(1-z))| + |\varphi_2(x(1-z))| \\ &\leq ||x||\,||\varphi_1 - \psi_1|| \left(\varepsilon^{\frac{1}{2}} + 2\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} + 2\varepsilon^{\frac{1}{2}}\right) \\ &= ||x||\,||\varphi_1 - \psi_1||6\varepsilon^{\frac{1}{2}}.\end{aligned}$$

As ε was arbitrary $\varphi_1 = \varphi_2$ and hence $\psi_1 = \psi_2$. □

We are now ready to reap the reward of our previous work.

Proposition 3.21 ([CGS+23](2.1))**.** *Let A be a unital C^*-algebra.*

(i) If $f \in \mathrm{Aff}T(A)$, then we can pick $a \in A_{sa}$ such that $f = \widehat{a}$, where $\widehat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$. Further a is unique modulo $\overline{[A, A]}$ and given any $\varepsilon > 0$ we can choose a such that $||a|| < ||f|| + \varepsilon$.

(ii) $[A, A] \cap A_{sa}$ is spanned by $\{[x, x^*] \mid x \in A\}$.

Proof. Before showing property (i) and (ii), we show the space of bounded tracial functionals $T_{\mathbb{C}}(A) := \{\tau \in A^* | \tau|_{[A,A]} = 0\}$ is canonically isomorphic to $(A/\overline{[A, A]})^*$ such that the weak* topology is preserved, then for $f \in \mathrm{Aff}T(A)$ we extend it to a w^*-continuous self-adjoint bounded linear functional $\widetilde{f} : T_{\mathbb{C}}(A) \to \mathbb{C}$. We show $T_{\mathbb{C}}(A)$ and $(A/\overline{[A, A]})^*$ are isomorphic.
Let $\varphi : T_{\mathbb{C}}(A) \to (A/\overline{[A, A]})^*$ be given by $\varphi(\tau) = [\tau]_{A/\overline{[A,A]}}$, where $[\tau]_{A/\overline{[A,A]}}(a) = \tau([a])$. This is

a well defined map, and is bijective as $\psi \in (A/\overline{[A,A]})^*$ are just the bounded linear functionals mapping elements of $\overline{[A,A]}$ to 0. Clearly φ is linear, and an isometry, here the norms are the operator norms on $T_{\mathbb{C}}(A)$ and $(A/\overline{[A,A]})^*$ respectively. We show φ preserves the weak* topology, that is we show it is continuous with continuous inverse. Let $(\tau_i)_{i\in I} \subset T_{\mathbb{C}}(A)$, $\tau_i \to \tau \in T_{\mathbb{C}}(A)$ be a converging net,

$$\varphi(\tau_i)(a) = [\tau_i]_{A/\overline{[A,A]}}(a) = \tau_i([a]) \to \tau([a]) = \varphi(\tau)(a),$$

hence φ is weak* continuous. Let $([\tau_j]_{A/\overline{[A,A]}})_{j\in J} \subset (A/\overline{[A,A]})^*$, $[\tau_j]_{A/\overline{[A,A]}} \to [\tau]_{A/\overline{[A,A]}}$ be a converging net,

$$\varphi^{-1}([\tau_j]_{A/\overline{[A,A]}})(a) = \tau_j(a) \to \tau(a) = \varphi^{-1}([\tau]_{A/\overline{[A,A]}})(a).$$

We show there exists an extension of $f \in \mathrm{Aff}T(A)$ to a w^*-continuous self-adjoint bounded linear functional $\widetilde{f} : T_{\mathbb{C}}(A) \to \mathbb{C}$. First define $\overline{f} : T_+(A) \to \mathbb{R}$ from the positive traces to the real numbers by $\overline{f}(\tau) = \|\tau\| f\left(\frac{\tau}{||\tau||}\right)$. For the next bit let $\tau \in T_{\mathbb{C}}(A)$, and decompose τ into self-adjoint functionals

$$\tau = \frac{1}{2}(\tau + \tau^*) + i\frac{1}{2i}(\tau - \tau^*),$$

we will denote $\frac{1}{2}(\tau+\tau^*)$ by $\mathrm{Re}(\tau)$ and $\frac{1}{2i}(\tau-\tau^*)$ by $\mathrm{Im}(\tau)$ and call them the real and imaginary part respectively. Note $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ are selfadjoint and $\mathbb{R}$-linear. Since they are self-adjoint we find their Jordan decomposition, $\mathrm{Re}(\tau)_\pm$ and $\mathrm{Im}(\tau)_\pm$. Define $\widetilde{f} : T_{\mathbb{C}}(A) \to \mathbb{C}$ by

$$\widetilde{f}(\tau) = \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + i(\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-)).$$

We show $\widetilde{f}$ is self-adjoint. By Definition 3.17

$$\begin{aligned}\widetilde{f}^*(\tau) = \overline{\widetilde{f}(\tau^*)} &= \overline{\overline{f}(\mathrm{Re}(\tau^*)_+) - \overline{f}(\mathrm{Re}(\tau^*)_-) + i(\overline{f}(\mathrm{Im}(\tau^*)_+) - \overline{f}(\mathrm{Im}(\tau^*)_-))} \\ &= \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) - i(\overline{f}((-\mathrm{Im}(\tau))_+) - \overline{f}((-\mathrm{Im}(\tau))_-)).\end{aligned}$$

Since $(-\tau)_+ = \tau_-$ and $(-\tau)_- = \tau_+$, which follows from uniqueness of the Jordan decomposition,

$$\begin{aligned}&= \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) - i(\overline{f}(\mathrm{Im}(\tau)_-) - \overline{f}(\mathrm{Im}(\tau)_+)) \\ &= \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + i(\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-)) = \widetilde{f}(\tau).\end{aligned}$$

We show linearity of $\widetilde{f}$, let $\sigma \in T_{\mathbb{C}}(A)$. First we show additivity of $\widetilde{f}$, to see this we need additivity of $\overline{f}$. Observe

$$\overline{f}(\tau+\sigma) = ||\tau+\sigma|| f\left(\frac{\tau+\sigma}{||\tau+\sigma||}\right) = ||\tau+\sigma|| f\left(\frac{\tau+\sigma}{||\tau||+||\sigma||}\right),$$

where the last equality comes from that the norm is additive on positive linear functionals. Note

$$\frac{||\tau||}{||\tau||+||\sigma||}\frac{\tau}{||\tau||} + \left(1 - \frac{||\tau||}{||\tau||+||\sigma||}\right)\frac{\sigma}{||\sigma||} = \frac{\tau}{||\tau||+||\sigma||} + \frac{||\sigma||}{||\tau||+||\sigma||}\frac{\sigma}{||\sigma||} = \frac{\tau+\sigma}{||\tau||+||\sigma||}.$$

Using f is affine

$$\begin{aligned}||\tau+\sigma||\left(\frac{||\tau||}{||\tau+\sigma||} f\left(\frac{\tau}{||\tau||}\right) + \frac{||\sigma||}{||\tau+\sigma||} f\left(\frac{\sigma}{||\sigma||}\right)\right) &= ||\tau|| f\left(\frac{\tau}{||\tau||}\right) + \|\sigma\| f\left(\frac{\sigma}{||\sigma||}\right) \\ &= \overline{f}(\tau) + \overline{f}(\sigma).\end{aligned}$$

Let $c \in \mathbb{R}^+$ then $\overline{f}(c\tau) = ||c\tau|| f\left(\frac{c\tau}{||c\tau||}\right) = c\overline{f}(\tau)$, and hence $\overline{f}$ is $\mathbb{R}^+$-linear. Observe for φ some self-adjoint linear functional, and $\psi_1, \psi_2 \geq 0$ be positive linear functionals such that $\varphi_+ - \varphi_- = \psi_1 - \psi_2$ then $\overline{f}(\varphi_+) - \overline{f}(\varphi_-) = \overline{f}(\psi_1) - \overline{f}(\psi_2)$ indeed

$$\overline{f}(\psi_1) + \overline{f}(\varphi_-) = \overline{f}(\psi_1 + \varphi_-) = \overline{f}(\varphi_+ + \psi_2) = \overline{f}(\varphi_+) + \overline{f}(\psi_2).$$

Recalling that $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ was $\mathbb{R}$ linear and self-adjoint

$$\begin{aligned}\mathrm{Re}(\tau+\sigma)_+ - \mathrm{Re}(\tau+\sigma)_- = \mathrm{Re}(\tau+\sigma) &= \mathrm{Re}(\tau) + \mathrm{Re}(\sigma)\\ &= \mathrm{Re}(\tau)_+ - \mathrm{Re}(\tau)_- + \mathrm{Re}(\sigma)_+ - \mathrm{Re}(\sigma)_-.\end{aligned}$$

Hence

$$\overline{f}(\mathrm{Re}(\tau+\sigma)_+) - \overline{f}(\mathrm{Re}(\tau+\sigma)_-) = \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + \overline{f}(\mathrm{Re}(\sigma)_+) - \overline{f}(\mathrm{Re}(\sigma)_-),$$

analogously one observes

$$\overline{f}(\mathrm{Im}(\tau+\sigma)_+) - \overline{f}(\mathrm{Im}(\tau+\sigma)_-) = \overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-) + \overline{f}(\mathrm{Im}(\sigma)_+) - \overline{f}(\mathrm{Im}(\sigma)_-).$$

Using this

$$\begin{aligned}\widetilde{f}(\tau+\sigma) &= \overline{f}(\mathrm{Re}(\tau+\sigma)_+) - \overline{f}(\mathrm{Re}(\tau+\sigma)_-) + i(\overline{f}(\mathrm{Im}(\tau+\sigma)_+) - \overline{f}(\mathrm{Im}(\tau+\sigma)_-))\\ &= \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + \overline{f}(\mathrm{Re}(\sigma)_+) - \overline{f}(\mathrm{Re}(\sigma)_-)\\ &\quad + i(\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-) + \overline{f}(\mathrm{Im}(\sigma)_+) - \overline{f}(\mathrm{Im}(\sigma)_-)\\ &= \widetilde{f}(\tau) + \widetilde{f}(\sigma).\end{aligned}$$

Let $\alpha \in \mathbb{C}$, $\alpha = a + ib$ for $a, b \in \mathbb{R}$, then

$$\widetilde{f}(\alpha\tau) = \widetilde{f}(a\tau) + \widetilde{f}(ib\tau).$$

Consider $\widetilde{f}(a\tau)$, if $a > 0$ then $(a\tau)_+ = a\tau_+$ and $(a\tau)_- = a\tau_-$, and as $\overline{f}$ was $\mathbb{R}^+$-linear and $\mathrm{Re}(\tau), \mathrm{Im}(\tau)$ are both $\mathbb{R}$-linear

$$\begin{aligned}\widetilde{f}(a\tau) &= \overline{f}(\mathrm{Re}(a\tau)_+) - \overline{f}(\mathrm{Re}(a\tau)_-) + i(\overline{f}(\mathrm{Im}(a\tau)_+) - \overline{f}(\mathrm{Im}(a\tau)_-))\\ &= a\overline{f}(\mathrm{Re}(\tau)_+) - a\overline{f}(\mathrm{Re}(\tau)_-) + i(a\overline{f}(\mathrm{Im}(\tau)_+) - a\overline{f}(\mathrm{Im}(\tau)_-))\\ &= a\widetilde{f}(\tau).\end{aligned}$$

If $a < 0$ then

$$\begin{aligned}\widetilde{f}(a\tau) &= \widetilde{f}(-(-a)\tau)\\ &= -a\widetilde{f}(-\tau)\\ &= -a(\overline{f}(\mathrm{Re}(-\tau)_+) - \overline{f}(\mathrm{Re}(-\tau)_-) + i(\overline{f}(\mathrm{Im}(-\tau)_+) - \overline{f}(\mathrm{Im}(-\tau)_-)))\\ &= -a(\overline{f}(\mathrm{Re}(\tau)_-) - \overline{f}(\mathrm{Re}(\tau)_+) + i(\overline{f}(\mathrm{Im}(\tau)_-) - \overline{f}(\mathrm{Im}(\tau)_+)))\\ &= a(\overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + i(\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-))\\ &= a\widetilde{f}(\tau).\end{aligned}$$

Lastly if $a = 0$ then $\widetilde{f}(a\tau) = a\widetilde{f}(\tau)$. Hence

$$\widetilde{f}(a\tau) + \widetilde{f}(ib\tau) = a\widetilde{f}(\tau) + b\widetilde{f}(i\tau).$$

We show $\widetilde{f}(i\tau) = i\widetilde{f}(\tau)$. Note

$$\begin{aligned}\mathrm{Re}(i\tau) &= \frac{1}{2}(i\tau + (i\tau)^*) = \frac{i}{2}(\tau - \tau^*) = -\frac{1}{2i}(\tau - \tau^*) = -\mathrm{Im}(\tau),\\ \mathrm{Im}(i\tau) &= \frac{1}{2i}(i\tau - (i\tau)^*) = \frac{1}{2}(\tau + \tau^*) = \mathrm{Re}(\tau).\end{aligned}$$

Using this

$$\begin{aligned}\widetilde{f}(i\tau) &= \overline{f}(\mathrm{Re}(i\tau)_+) - \overline{f}(\mathrm{Re}(i\tau)_-) + i(\overline{f}(\mathrm{Im}(i\tau)_+) - \overline{f}(\mathrm{Im}(i\tau)_-))\\ &= \overline{f}((-\mathrm{Im}(\tau))_+) - \overline{f}((-\mathrm{Im}(\tau))_-) + i(\overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-))\\ &= i(\overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-)) + \overline{f}(\mathrm{Im}(\tau)_-) - \overline{f}(\mathrm{Im}(\tau)_+)\\ &= i(\overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-)) - (\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-))\\ &= i\widetilde{f}(\tau).\end{aligned}$$

We show $||\widetilde{f}|| = ||f||$, hence $\widetilde{f}$ is bounded. For all $\tau \in T_{\mathbb{C}}(A)$ there exists a $z \in \mathbb{C}$ with $|z| = 1$ such that $|\widetilde{f}(\tau)| = z\widetilde{f}(\tau) = \widetilde{f}(z\tau)$. Let $\varepsilon > 0$ then there exists $\tau \in T_{\mathbb{C}}(A)$ such that

$$||\widetilde{f}|| \leq |\widetilde{f}(\tau)| + \varepsilon = \widetilde{f}(z\tau) + \varepsilon.$$

As $\widetilde{f}(z\tau) = |\widetilde{f}(\tau)|$ we must have $\widetilde{f}(z\tau) \in \mathbb{R}$, continuing our calculation

$$= \frac{1}{2}\left(\widetilde{f}(z\tau) + \overline{\widetilde{f}(z\tau)}\right) + \varepsilon = \widetilde{f}\left(\frac{1}{2}(z\tau + (z\tau)^*\right) + \varepsilon = \widetilde{f}(\mathrm{Re}(z\tau)) + \varepsilon.$$

Observe $||\mathrm{Re}(z\tau)|| \leq ||z\tau|| = ||\tau||$. If $||\tau|| \leq 1$ then $||\mathrm{Re}(z\tau)|| \leq 1$, and as $\mathrm{Re}(z\tau)$ is self-adjoint then

$$||\widetilde{f}|| = \sup_{\substack{\tau \in T_{\mathbb{C}}(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)| \leq \sup_{\substack{\tau \in T_{\mathbb{R}}(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)|.$$

Further as $T_{\mathbb{R}}(A) \subseteq T_{\mathbb{C}}(A)$, $\sup_{\substack{\tau \in T_{\mathbb{R}}(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)| \leq \sup_{\substack{\tau \in T_{\mathbb{C}}(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)|$ hence

$$||\widetilde{f}|| = \sup_{\substack{\tau \in T_{\mathbb{R}}(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)|.$$

Therefore, to compute $\|\widetilde{f}\|$ it is enough to consider $\tau \in T_{\mathbb{R}}(A)$. We find the Jordan decomposition of τ, $\tau = \tau_+ - \tau_-$, and $||\tau_+|| + ||\tau_-|| = ||\tau||$. If $||\tau|| \leq 1$ then $||\tau_+ + \tau_-|| \leq ||\tau|| \leq 1$. Pick $\tau \in T_{\mathbb{R}}(A)$ such that

$$||\widetilde{f}|| \leq |\widetilde{f}(\tau)| + \varepsilon = |\widetilde{f}(\tau_+ - \tau_-)| + \varepsilon \leq |\widetilde{f}(\tau_+)| + |\widetilde{f}(\tau_-)| + \varepsilon.$$

As τ_+, τ_- are positive, we get that $\widetilde{f}(\tau_+), \widetilde{f}(\tau_-)$ are positive numbers, hence

$$|\widetilde{f}(\tau_+)| + |\widetilde{f}(\tau_-)| + \varepsilon = \widetilde{f}(\tau_+ + \tau_-) + \varepsilon.$$

Since $\tau_+ + \tau_-$ are positive elements with norm less than or equal to $||\tau||$,

$$||\widetilde{f}|| \leq \sup_{\substack{\tau \in T_+(A)\\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)|,$$

and as $T_+(A) \subseteq T_{\mathbb{R}}(A)$

$$||\widetilde{f}|| = \sup_{\substack{\tau \in T_+(A) \\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)|.$$

If $T_+(A)$ is not $\{0\}$ then

$$\sup_{\substack{\tau \in T_+(A) \\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)| = \sup_{\substack{\tau \in T_+(A) \\ ||\tau|| = 1}} |\widetilde{f}(\tau)| = ||f||.$$

If $T_+(A) = \{0\}$, then trivially

$$\sup_{\substack{\tau \in T_+(A) \\ ||\tau|| \leq 1}} |\widetilde{f}(\tau)| = \sup_{\tau \in T_{(A)}} |\widetilde{f}(\tau)| = \|f\|.$$

We show $\widetilde{f}$ is w^*-continuous. By Corollary 2.7.9 in [Meg98] to show w^*-continuity of $\widetilde{f}$ it suffices to show $\widetilde{f}$ is w^*-continuous on the unit ball of $T_{\mathbb{C}}(A)$, which we will denote by $T^1_{\mathbb{C}}(A)$. Let $(\tau_i)_{i \in I} \subset T^1_{\mathbb{C}}(A)$ be a converging net $\tau_i \to \tau$. Then

$$\widetilde{f}(\tau_i) = \overline{f}(\mathrm{Re}(\tau_i)_+) - \overline{f}(\mathrm{Re}(\tau_i)_-) + i(\overline{f}(\mathrm{Im}(\tau_i)_+) + \overline{f}(\mathrm{Im}(\tau_i)_-)).$$

By the construction of the Jordan decomposition

$$\mathrm{Re}(\tau_i)_+, \mathrm{Re}(\tau_i)_-, \mathrm{Im}(\tau_i)_+, \mathrm{Im}(\tau_i)_- \in T^1_{\mathbb{C}}(A).$$

Hence

$$(\mathrm{Re}(\tau_i)_+, \mathrm{Re}(\tau_i)_-, \mathrm{Im}(\tau_i)_+, \mathrm{Im}(\tau_i)_-)_{i \in I} \subset \left(T^1_{\mathbb{C}}(A)\right)^4$$

induces a net. As $T^1_{\mathbb{C}}(A)$ is compact then $(T^1_{\mathbb{C}}(A))^4$ is compact so there exists a subnet $(\tau_{i_j})_{i_j \in I}$ such that

$$\mathrm{Re}(\tau_{i_j})_+ \to \eta_1,\ \mathrm{Re}(\tau_{i_j})_- \to \eta_2,\ \mathrm{Im}(\tau_{i_j})_+ \to \eta_3,\ \mathrm{Im}(\tau_{i_j})_- \to \eta_4.$$

Further as $\mathrm{Re}(\tau_{i_j}) \to \mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau_{i_j}) \to \mathrm{Im}(\tau)$, and the w^* topology is Hausdorff

$$\eta_1 - \eta_2 = \mathrm{Re}(\tau)_+ - \mathrm{Re}(\tau)_-, \quad \eta_3 - \eta_4 = \mathrm{Im}(\tau)_+ - \mathrm{Im}(\tau)_-.$$

Recall f is norm continuous, and hence w^* continuous, thus $\overline{f}$ will also be w^* continuous on positive functionals. Hence

$$\begin{aligned}\widetilde{f}(\tau_{i_j}) &= \overline{f}(\mathrm{Re}(\tau_{i_j})_+) - \overline{f}(\mathrm{Re}(\tau_{i_j})_-) + i(\overline{f}(\mathrm{Im}(\tau_{i_j})_+) + \overline{f}(\mathrm{Im}(\tau_{i_j})_-)) \\ &\to \overline{f}(\eta_1) - \overline{f}(\eta_2) + i(\overline{f}(\eta_3) + \overline{f}(\eta_4)).\end{aligned}$$

As positive elements are a closed set the η's are positive and since

$$\eta_1 - \eta_2 = \mathrm{Re}(\tau)_+ - \mathrm{Re}(\tau)_-, \quad \eta_3 - \eta_4 = \mathrm{Im}(\tau)_+ - \mathrm{Im}(\tau)_-,$$

we get

$$\overline{f}(\eta_1) - \overline{f}(\eta_2) + i(\overline{f}(\eta_3) + \overline{f}(\eta_4) = \overline{f}(\mathrm{Re}(\tau)_+) - \overline{f}(\mathrm{Re}(\tau)_-) + i(\overline{f}(\mathrm{Im}(\tau)_+) - \overline{f}(\mathrm{Im}(\tau)_-)) = \widetilde{f}(\tau).$$

Thus for any convergent net, we have found a subnet such that $\widetilde{f}$ of the subnet is equal to $\widetilde{f}$ of the limit, luckily this is equivalent to normal continuity and hence we have shown that f extends to a w^*-continuous self-adjoint linear functional $\widetilde{f}$, thus $\widetilde{f} \in (T_{\mathbb{C}}(A))^* \cong (A/[A,A])^*$. Recall for any

Banach space, B, we can identify the double dual with B when B^* is equipped with the weak* topology, hence $A/\overline{[A,A]} \cong ((A/\overline{[A,A]}, ||\cdot||)^*, w^*)^* \cong (T_{\mathbb{C}}(A))^*$, where the first isomorphism is identifying with evaluation functionals.
We prove (i). Let $f \in \mathrm{Aff}T(A)$ and extend f to $\widetilde{f} \in (T_{\mathbb{C}}(A))^*$. Then there exists $a \in A$ such that $[a]_{A/\overline{[A,A]}} \mapsto \widetilde{f}$ using our above two isomorphisms, further as $\widetilde{f}$ is self-adjoint we can find a self-adjoint. Uniqueness follows as we are using isomorphisms. As $\widetilde{f}$ is weak* continuous this yields that $\widetilde{f} = \widehat{a}$. Let $\varepsilon > 0$, and recall our isomorphism is an isometry so $||[a]||_{A/\overline{[A,A]}} = ||\widetilde{f}|| = ||f||$ and by definition of the quotient norm

$$||a|| \leq ||[a]||_{A/\overline{[A,A]}} + \varepsilon = ||f|| + \varepsilon.$$

We show (ii). Note

$$[x, x^*]^* = (xx^* - x^*x)^* = xx^* - x^*x = [x, x^*].$$

Hence $\mathrm{span}_{\mathbb{R}}([x, x^*] \mid x \in A) \subseteq A_{sa}$ and so $\mathrm{span}_{\mathbb{R}}([x, x^*]) \subseteq [A, A] \cap A_{sa}$. Let $a, b \in A$ we apply the polarisation identity on $[a, b^*]$

$$\begin{aligned}[a, b^*] = &\left[\frac{1}{2}(a+b), \frac{1}{2}(a+b)^*\right] - \left[\frac{1}{2}(a-b), \frac{1}{2}(a-b)^*\right] \\ &+ i\left[\frac{1}{2}(a+ib), \frac{1}{2}(a+ib)^*\right] - i\left[\frac{1}{2}(a-ib), \frac{1}{2}(a-ib)^*\right].\end{aligned}$$

To simplify notation set

$$\begin{aligned} x_1 &= \frac{1}{2}(a+b), \\ x_2 &= \frac{1}{2}(a-b), \\ x_3 &= \frac{1}{2}(a+ib), \\ x_4 &= \frac{1}{2}(a-ib). \end{aligned}$$

If a, b are self-adjoint then

$$[x_1, x_1^*] - [x_2, x_2^*] + i[x_3, x_3^*] - i[x_4, x_4^*] = [x_1, x_1^*] - [x_2, x_2^*] - i[x_3, x_3^*] + i[x_4, x_4^*],$$

hence $[x_3, x_3^*] = [x_4, x_4^*]$. For $h \in [A, A] \cap A_{sa}$ write

$$h = \sum_j \lambda_j [a_j, b_j^*],$$

where $a_j, b_j^* \in A$. Since $h \in A_{sa}$, $[a_j, b_j]^* = [a_j, b_j^*]$ and $\lambda_j \in \mathbb{R}$. Applying the polarisation identity to h yields

$$h = \sum_j \lambda_j [a_j, b_j^*] = \sum_j \lambda_j([x_{j,1}, x_{j,1}^*] - [x_{j,2}, x_{j,2}^*] \in \mathrm{span}_{\mathbb{R}}([x, x^*] \mid x \in A).$$

□

Our next goal is to show $S(\mathrm{Aff}T(A)) \cong T(A)$ and apply Kadison duality, however we need to how we extend maps defined on self-adjoint elements to the entire C^*-algebra.

Lemma 3.22. *Let A be a unital C^*-algebra and let $\varphi : A_{sa} \to \mathbb{R}$ be a bounded linear functional. Then there exists a self-adjoint bounded linear functional $\widetilde{\varphi} : A \to \mathbb{C}$ such that*

$$\widetilde{\varphi}(x) = \varphi(a) + i\varphi(b), \quad a, b \in A_{sa}.$$

Moreover if $\varphi \in T(A)$ then $\widetilde{\varphi}$ is a tracial state.

Proof. Let $x \in A$, and decompose $x = a + ib$ where $a, b \in A_{sa}$. Hence defining $\widetilde{\varphi}$ by

$$\widetilde{\varphi}(x) := \widetilde{\varphi}(a + ib) = \varphi(a) + i\varphi(b)$$

makes sense. Observe that $\widetilde{\varphi}$ is linear and bounded since φ is. Assume $x \in A_{sa}$, then the decomposition implies $x = a$, thus $\widetilde{\varphi}$ extends φ. Assume $\varphi \in T(A)$. We want to show that $\widetilde{\varphi}$ has the trace property and is a state. As positive elements are self-adjoint applying $\widetilde{\varphi}$ is the same as applying φ thus $\widetilde{\varphi}$ is positive, and $\widetilde{\varphi}(1_A) = 1$, hence $\widetilde{\varphi}$ is a state. What is left to show is that $\widetilde{\varphi}$ has the trace property. Recall the polarisation identity

$$x^* y = \frac{1}{4} \sum_{k=0}^{3} i^k (i^k x + y)^* (i^k x + y),$$

and note the product in the sum is on the form $z^* z$ hence is a positive element. So

$$\begin{aligned}
\widetilde{\varphi}(x^* y) &= \frac{1}{4} \sum_{k=0}^{3} i^k \widetilde{\varphi}((i^k x + y)^* (i^k x + y)) \\
&= \frac{1}{4} \sum_{k=0}^{3} i^k \varphi((i^k x + y)^* (i^k x + y)) \\
&= \frac{1}{4} \sum_{k=0}^{3} i^k \varphi((i^k x + y)(i^k x + y)^*) \\
&= \frac{1}{4} \sum_{k=0}^{3} i^k \widetilde{\varphi}((i^k x + y)(i^k x + y)^*) \\
&= \widetilde{\varphi}(y x^*).
\end{aligned}$$

□

We are now ready to apply Kadison duality to $\mathrm{Aff}T(A)$.

Proposition 3.23. *Let A be a unital C^*-algebra and let $T(A)$ denote the space of all tracial states. Then the state space of $\mathrm{Aff}T(A)$ is affinely homeomorphic to $T(A)$. By Kadison duality $\mathrm{Aff}T(A)$ and $T(A)$ has the same data.*

Proof. To show $S(\mathrm{Aff}T(A)) \cong T(A)$ we construct two positive affine continuous maps, χ, θ, and show they are inverses. The two maps $\chi : S(\mathrm{Aff}T(A)) \to T(A)$ and $\theta : T(A) \to S(\mathrm{Aff}T(A))$ are given by

$$\chi(\varphi)(a) = \varphi(\widehat{a}) \quad \theta(\psi)(f) = f(\psi),$$

for $a \in A_{sa}$ and $f \in \mathrm{Aff}T(A)$. χ makes sense as by Lemma 3.22 it is enough to define $\chi(\varphi)$ on self-adjoint elements. For the rest the proof let $\varphi_1, \varphi_2 \in S(\mathrm{Aff}T(A))$, $\psi_1, \psi_2 \in T(A)$, $x \in A$ with decomposition $x = b_1 + ib_2$ for $b_1, b_2 \in A_{sa}$ and for $f \in \text{Aff}T(A)$ find $a \in A_{sa}$ such that $f = \widehat{a}$ by Proposition 3.21. We show χ and θ are inverses. First

$$\chi(\theta(\psi))(a) = \theta(\psi)(\widehat{a}) = \widehat{a}(\psi) = \psi(a),$$

then by definition of $\widetilde{\chi}$ and the above

$$\widetilde{\chi}(\theta(\psi))(x) = \chi(\theta(\psi))(b_1) + i\chi(\theta(\psi))(b_2) = \psi(b_1) + i\psi(b_2) = \psi(x).$$

The other composition

$$\theta(\chi(\varphi))(\widehat{a}) = \widehat{a}(\chi(\varphi)) = \chi(\varphi)(a) = \varphi(\widehat{a}).$$

Hence we have a bijection between the two spaces. We show positivity. First χ, let $\varphi_1 \leq \varphi_2$ then

$$\chi(\varphi_1)(a) = \varphi_1(\widehat{a}) \leq \varphi_2(\widehat{a}) = \chi(\varphi_2)(a).$$

Then $\widetilde{\chi}$

$$\widetilde{\chi}(\varphi_1)(x) = \chi(\varphi_1)(b_1) + i\chi(\varphi_1)(b_2) \leq \chi(\varphi_2)(b_1) + i\chi(\varphi_2)(b_2) = \widetilde{\chi}(\varphi_2)(x).$$

Now θ, let $\psi_1 \leq \psi_2$

$$\theta(\psi_1)(\widehat{a}) = \widehat{a}(\psi_1) = \psi_1(a) \leq \psi_2(a) = \theta(\psi_2)(a).$$

We show χ, θ are affine. First χ, let $t \in [0, 1]$,

$$\chi(t\varphi_1 + (1-t)\varphi_2)(a) = (t\varphi_1 + (1-t)\varphi_2)(\widehat{a}) = t\varphi_1(\widehat{a}) + (1-t)\varphi_2(\widehat{a}) = t\chi(\varphi_1)(a) + (1-t)\chi(\varphi_2)(a).$$

Then for $\widetilde{\chi}$

$$\begin{aligned}\widetilde{\chi}(t\varphi_1 + (1-t)\varphi_2)(x) &= t(\chi(\varphi_1)(b_1) + i\chi(\varphi_1)(b_2)) + (1-t)(\chi(\varphi_2)(b_1) + i\chi(\varphi_2)(b_2)) \\ &= t\widetilde{\chi}(\varphi_1)(x) + (1-t)\widetilde{\chi}(\varphi_2)(x)\end{aligned}$$

Now θ,

$$\theta(t\psi_1 + (1-t)\psi_2)(\widehat{a}) = \widehat{a}(t\psi_1 + (1-t)\psi_2) = t\widehat{a}(\psi_1) + (1-t)\widehat{a}(\psi_2) = t\theta(\psi_1)(\widehat{a}) + (1-t)\theta(\psi)(\widehat{a}).$$

We show continuity. Let $(\varphi_i)_{i\in I} \subset S(\text{Aff}T(A))$ and $(\psi_j)_{j\in J} \subset T(A)$ be converging nets

$$\varphi_i \to \varphi, \quad \psi_j \to \psi.$$

Both spaces has the w^* topology, hence the nets $(\varphi_i)_{i\in I}, (\psi_j)_{j\in J}$ converges if and only if for all $f \in \text{Aff}T(A)$ and for all $a \in A$

$$\varphi_i(f) \to \varphi(f), \quad \psi_i(a) \to \psi(a).$$

We show continuity for χ.

$$\chi(\varphi_i)(a) = \varphi_i(\widehat{a}) \to \varphi(\widehat{a}) = \chi(\varphi)(a),$$

Then for $\widetilde{\chi}$.

$$\widetilde{\chi}(\varphi_i)(x) = \chi(\varphi_i)(b_1) + i\chi(\varphi_i)(b_2) \to \chi(\varphi)(b_1) + i\chi(\varphi)(b_2) = \widetilde{\chi}(\varphi)(x).$$

Now θ.

$$\theta(\psi_i)(\widehat{a}) = \widehat{a}(\psi_i) = \psi_i(a) \to \psi(a) = \theta(\psi)(\widehat{a}).$$

Recalling that $T(A)$ is compact in the w^* topology $\text{Aff}T(A)$ is complete by Lemma 3.13, and by Kadison duality 3.15 $\text{Aff}T(A)$ and $S(\text{Aff}T(A)) \cong T(A)$ has the same data. $\square$

We are now done with this small detour into the world of order unit spaces, and can conclude that for a unital C^*-algebra A working with $T(A)$ or $\text{Aff}T(A)$ will give the same results by Kadison duality. Moving forward in this project we will consider $\text{Aff}T(A)$ instead of $T(A)$.

4 The functors KT_u and $\overline{K}_1^{\text{alg}}$

In this section we will introduce the pairing map, ρ_A, which connects $K_0(A)$ and $\text{Aff}T(A)$. Then we define two functors, $\overline{K}_1^{\text{alg}}(A)$ and $KT_u(A)$ both of which are constituents in the total invariant we will define in section 6.

4.1 KT_u

To define KT_u of a C^*-algebra we first need to define the pairing map.

Definition 4.1. Let A be a unital C^*-algebra and define the *paring map* $\rho_A : K_0(A) \to \text{Aff}T(A)$, for $p, q \in \mathcal{P}_n(A)$ and $\tau \in T(A)$ by

$$\rho_A([p]_0 - [q]_0)(\tau) := \tau_n(p - q),$$

where τ_n denotes the non-normalised extension of τ to $M_n(A)$ that is $\tau_n(1_{M_n(A)}) = n$. We write

$$(\widehat{p}_n - \widehat{q}_n)(\tau) = \tau_n(p - q).$$

for the non-normalised extension of the evaluation.

We now define one of the key objects in this thesis, KT_u, together with morphisms between such objects, both of which will be studied in more detail later in the thesis.

Definition 4.2 ([CGS$^+$23](2.3))**.** Let KT_u be the functor on the category of unital C^*-algebras with unital *-homomorphisms that assigns to a unital C^*-algebra the quadruple

$$KT_u(A) = (K_*(A), [1_A]_0, \text{Aff}T(A), \rho_A).$$

A KT_u morphism $(\alpha_*, \gamma) : KT_u(A) \to KT_u(B)$ consists of a pair $\alpha_* = (\alpha_0, \alpha_1)$ of homomorphisms $\alpha_i : K_i(A) \to K_i(B)$, for $i \in \{0, 1\}$ with $\alpha_0([1_A]_0) = [1_B]_0$ and a positive linear map $\gamma : \text{Aff}T(A) \to \text{Aff}T(B)$ such that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \text{Aff}T(A) \\ {\scriptstyle \alpha_0}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ K_0(B) & \xrightarrow[\rho_B]{} & \text{Aff}T(B) \end{array}$$

commutes.

We now move to defining the other functor, $\overline{K}_1^{\text{alg}}(A)$, which will require a bit more work.

4.2 Constructing $\overline{K}_1^{\text{alg}}$

For this section let A be a unital C^*-algebra and denote by $U_n(A)$ the group of unitaries in $M_n(A)$, in the norm topology. We will also denote by $DU_n(A)$ the derived subgroup of $U_n(A)$, that is elements on the form uvu^*v^* where $u, v \in U_n(A)$. We wish to define a direct limit of $U_n(A)$, so consider the connecting maps $\iota_{n,m} : U_n(A) \to U_m(A)$ given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

and construct the direct limit $U_\infty(A)$. We equip this with the inductive limit topology, which is given by $V \subseteq U_\infty(A)$ is open if and only if $V \cap U_n(A)$ is open for all $n \in \mathbb{N}$. We also define the infinite derived group, $DU_\infty := \bigcup_n DU_n(A)$. Note that in the inductive limit topology multiplication is separably continuous and inversion is continuous in $U_\infty(A)$ hence $\overline{DU_\infty(A)}$ will be a normal subgroup of $U_\infty(A)$ and therefore we $U_\infty(A)/\overline{DU_\infty(A)}$ is well defined.

Definition 4.3 ([CGS$^+$23](2.6)). Let A be a unital C^*-algebra. Define the Hausdorffized unitary algebraic K_1 group of A by

$$\overline{K}_1^{\text{alg}}(A) := U_\infty(A)/\overline{DU_\infty(A)}.$$

We write $[u]_{\text{alg}}$ for the equivalence class of $u \in U_\infty(A)$ in $\overline{K}_1^{\text{alg}}(A)$. Let $\varphi : A \to B$ be a unital *-homomorphism, then there is an induced map $\overline{K}_1^{\text{alg}}(\varphi) : \overline{K}_1^{\text{alg}}(A) \to \overline{K}_1^{\text{alg}}(B)$ given by $\overline{K}_1^{\text{alg}}(\varphi)([u]_{\text{alg}}) := [\varphi^{(n)}(u)]_{\text{alg}}$, for $u \in U_n(A)$, where $\varphi^{(n)}$ is the induced map on the matrix algebra $U_n(A)$. Hence $\overline{K}_1^{\text{alg}}$ is a functor from unital C^*-algebras to abelian groups.

One property of $\overline{DU_\infty(A)}$ is that all elements in it are connected to the identity.

Proposition 4.4. *Let A be a unital C^*-algebra and $\overline{DU_\infty(A)}$ its infinite derived subgroup, then $\overline{DU_\infty(A)} \subseteq U_\infty^{(0)}(A)$, where $U_n^{(0)}(A)$ is the unitaries homotopic to the identity.*

Proof. Let $u, v \in U_n(A)$, then

$$\begin{pmatrix} uvu^*v^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & u^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^* \end{pmatrix} \begin{pmatrix} u^* & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v \end{pmatrix}.$$

Thus to show that $uvu^*v^* \sim_h 1$ in $DU_\infty(A)$ it is enough to show that each of the four constituents in the product is connected to the identity. By Whitehead's lemma [RLL00](2.1.5)

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} u^* & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} v^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v^* & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 & 0 \\ 0 & v^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^* \end{pmatrix}.$$

Thus

$$\begin{pmatrix} uvu^*v^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

What we just saw implies there is a canonical projection of $\overline{K}_1^{\text{alg}}(A)$ unto $K_1(A)$.

Definition 4.5. [CGS$^+$23](2.9) Let A be a unital C^*-algebra. There exists a canonical surjection $\not{a}_A : \overline{K}_1^{\text{alg}}(A) \to K_1(A)$ given by $\not{a}_A([u]_{\text{alg}}) = [u]_1$ for $u \in U_\infty(A)$. This is well defined by Proposition 4.4.

This finishes what is a pretty small section, however the definitions, especially KT_u will be used later.

5 The Thomsen map

As the name suggest, this section will be devoted to the Thomsen map, which connects $\mathrm{Aff}T(A)$ and $\overline{K}_1^{\mathrm{alg}}$.

5.1 Constructing the Thomsen map

Our goal is to define a map, $\mathrm{Th}_A : \mathrm{Aff}T(A) \to \overline{K}_1^{\mathrm{alg}}(A)$, given by $\mathrm{Th}_A(\widehat{a}) = [e^{2\pi i a}]_{\mathrm{alg}}$ for some $a \in A_{sa}$ where A is a unital C^*-algebra. Recall by Proposition 3.21 any $f \in \mathrm{Aff}T(A)$ can be realized as the point evaluation functional $\widehat{a}$ for some self-adjoint element $a \in A_{sa}$, so it makes sense to define Th_A only for point evaluations of self-adjoint elements.

Proposition 5.1 ([CGS+23])**.** *Let A be a unital C^*-algebra and define $\mathrm{Th}_A : \mathrm{Aff}T(A) \to \overline{K}_1^{\mathrm{alg}}(A)$ by $\mathrm{Th}_A(\widehat{a}) = [e^{2\pi i a}]_{alg}$ using functional calculus. Then Th_A is a well defined continuous group homomorphism. We will call Th_A the Thomsen map.*

Proof. We check Th_A is a continuous group homomorphism. Recall working in $U_\infty(A)/DU_\infty(A)$ means that elements commute hence

$$e^{ia}e^{ib}(e^{-i\frac{a}{n}}e^{-i\frac{b}{n}})^n = e^{ia}e^{ib}e^{-ia}e^{-ib} = 1_A \mod DU_\infty(A).$$

Using the above together with the Lie-Trotter formula:

$$e^{ia}e^{ib}e^{-i(a+b)} = \lim_{n\to\infty} e^{ia}e^{ib}(e^{-i\frac{a}{n}}e^{-i\frac{b}{n}})^n,$$

we get

$$e^{ia}e^{ib}e^{-i(a+b)} = 1_A \mod \overline{DU}_\infty(A).$$

Hence $a \mapsto [e^{2\pi i a}]_{\mathrm{alg}}$ is a continuous group homomorphism. We show Th_A is well defined. By Proposition 3.21 we have $[A, A] \cap A_{sa} = \mathrm{span}\{[x, x^*] \mid x \in A\}$ and since we are working mod $DU_\infty(A)$ together with Th_A being a continuous group homomorphism it suffices to show

$$[e^{2\pi i(v^*v - vv^*)}]_{\mathrm{alg}} = 0,$$

for each $v \in A$. Pick some $y \in \mathbb{R}$ such that $||v|| < y$. For $t \in [0, 1]$ let $w_t = tv + y1_A$, we will often abuse notation and simply write $w_t = tv + y$. Note

$$\|y1_A - (tv + y1_A)\| = \|tv\| < y = \|(y1_A)^{-1}\|^{-1},$$

by Proposition 2.1.11 in [RLL00] w_t is invertible, and by Proposition 2.1.8 also in [RLL00] there exists a unitary, u_t such that $u_t = w_t|w_t|^{-1} = w_t(w_t^*w_t)^{-\frac{1}{2}}$, defined by functional calculus. Consider the element $w_1 = v + y$, we observe two properties of this element. First

$$\begin{aligned}
w_1^*w_1 - w_1w_1^* &= (v^* + y)(v + y) - (v + y)(v^* + y) \\
&= v^*v + yv^* + yv + y^2 - vv^* - yv - yv^* - y^2 \\
&= v^*v - vv^*.
\end{aligned}$$

To show the other property, note $w_1 = u_1|w_1|$ and $|w_1|$ is selfadjoint since $w_1^*w_1$ is selfadjoint.

$$\begin{aligned}
u_1w_1^*w_1u_1^* &= u_1|w_1|^*u_1^*u_1|w_1|u_1^* \\
&= w_1|w_1|^{-1}|w_1|^*|w_1|(|w_1|^{-1})^*w_1^* \\
&= w_1w_1^*.
\end{aligned}$$

Combining these properties

$$\begin{aligned} e^{2\pi i(v^*v-vv^*)} &= e^{2\pi i(w_1^*w_1-w_1w_1^*)} \mod DU(A) \\ &= e^{2\pi i(w_1^*w_1-u_1w_1^*w_1u_1^*)} \mod DU(A) \\ &= e^{2\pi i w_1^*w_1}e^{-2\pi i u_1w_1^*w_1u_1^*} \mod DU(A) \\ &= e^{2\pi i w_1^*w_1}u_1e^{-2\pi i w_1^*w_1}u_1^* \mod DU(A), \end{aligned}$$

where the last equality comes from functional calculus. Thus $e^{2\pi i(v^*v-vv^*)}$ is on the form aba^*b^* and hence is an element of $DU(A)$, therefore $[e^{2\pi i(v^*v-vv^*)}]_{\text{alg}} = 0$. □

We wish to construct a short exact sequence using the Thomsen map and the pairing map. However to show exactness we first need to consider another map entirely, namely de la Harpe - Skandalis determinant map.

5.2 de la Harpe-Skandalis map

In this section A will be a unital C^*-algebra and $U_n(A)$ will denote the group of unitaries in $M_n(A)$. As the section name suggest we first define the de la Harpe-Skandalis determinant map.

Definition 5.2 ([CGS+23][Proposition 2.11]). Let $u : [0,1] \to U_n(A)$ be a piecewise smooth path. For any trace on A, $\tau \in T(A)$, we define the de la Harpe-Skandalis determinant map by

$$\widetilde{\Delta}_A(u)(\tau) = \frac{1}{2\pi i}\int_0^1 \tau_n\left(\left(\frac{d}{dt}u(t)\right)u(t)^*\right)dt,$$

where τ_n is the non-normalised canonical extension of τ to $M_n(A)$.

There is a lot to unpack in this definition, does $\frac{d}{dt}u(t)$ even make sense? We will unravel the definition starting with the simple case, what happens when $u(t) = e^{2\pi ita}$ for some self-adjoint element $a \in A_{\text{sa}}$.

Proposition 5.3. *Let $a \in A_{sa}$ be a self-adjoint element of a unital C^*-algebra A. For each $t \in [0,1]$ define the element $e^{2\pi ita}$ by functional calculus. Then $u(t) = e^{2\pi ita}$ is a smooth path of unitaries and $\frac{d}{dt}u(t) = 2\pi iae^{2\pi ita}$.*

Proof. By construction $u(t)$ is a path of unitaries, we now compute $\frac{d}{dt}u(t)$. Fix $t_0 \in [0,1]$, since the spectrum of a is compact

$$\left(\frac{1}{t-t_0}\left(e^{2\pi itx}-e^{2\pi it_0x}\right)\right)_{t\in\mathbb{R}\setminus\{t_0\}},$$

will converge uniformly to $2\pi ixe^{2\pi it_0x}$ when x is in the spectrum of a for t going to t_0. As $e^{2\pi ita}$ is defined by functional calculus

$$\lim_{t\to t_0}\left\|\frac{1}{t-t_0}\left(e^{2\pi ita}-e^{2\pi it_0a}\right)-2\pi iae^{2\pi it_0a}\right\| = 0,$$

by definition $\frac{d}{dt}e^{2\pi ita} = 2\pi iae^{2\pi ita}$ which implies $e^{2\pi ita}$ is a smooth path of unitaries. □

As we have found $\frac{d}{dt}e^{2\pi ita}$ we can compute $\widetilde{\Delta}_A(u)$.

Example 5.4. Let $a \in A_{\mathrm{sa}}$ be a self-adjoint element of a unital C^*-algebra, A, and consider the smooth path of unitaries given by $u(t) = e^{2\pi ita}$. Then $\widetilde{\Delta}_A(u)(\tau) = \tau(a)$.

Proof. By Proposition 5.3

$$\widetilde{\Delta}_A(u) = \frac{1}{2\pi i}\int_0^1 \tau\left(2\pi i a e^{2\pi ita} e^{-2\pi ita}\right) dt = \int_0^1 \tau(a)dt = \tau(a).$$

□

This example will form the basis of how we compute the determinant of an arbitrary path. We will show later that any piecewise smooth path can be written in some exponential form and hence be calculated more or less as in the example.

Lemma 5.5. *Let* $\arccos : \left[-\frac{1}{2}, \frac{1}{2}\right] \to [0, \pi]$ *be the inverse function of* $\cos|_{[0,\pi]}^{\left[-\frac{1}{2},\frac{1}{2}\right]}$. *Then the function* $f : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{T}\backslash\{-1\}$ *given by* $f(t) = e^{2\pi it}$ *has inverse* $\frac{1}{2\pi i}\ln : \mathbb{T}\backslash\{-1\} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$ *which satisfies:*

- $\frac{1}{2\pi i}\ln$ *is smooth,*
- $|\frac{1}{2\pi i}\ln(z)| = \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}|z-1|^2\right)$.

Proof. f is smooth and for all $n \in \mathbb{N}_0$, $f^{(n)}(t) = (2\pi i)^n e^{2\pi it} \neq 0$, by the inverse function theorem $\frac{1}{2\pi i}\ln$ is smooth. The other property follows from a straight forward computation. Recall for $z \in \mathbb{T}\backslash\{-1\}$ we can write $z = e^{2\pi it} = \cos(2\pi t) + i\sin(2\pi t)$ for some $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, thus

$$\left|\frac{1}{2\pi i}\ln(z)\right| = |t| = \frac{1}{2\pi}\arccos(\cos(2\pi t)) = \frac{1}{2\pi}\arccos(\mathrm{Re}(z)).$$

One observes

$$\begin{aligned}|z-1|^2 &= (\cos(2\pi t) - 1)^2 + \sin^2(2\pi t)\\ &= 1 + \cos^2(2\pi t) - 2\cos(2\pi t) + \sin^2(2\pi t)\\ &= 2(1 - \cos(2\pi t))\\ &= 2(1 - \mathrm{Re}(z)),\end{aligned}$$

so $\mathrm{Re}(z) = 1 - \frac{1}{2}|z-1|^2$. Hence

$$\left|\frac{1}{2\pi i}\ln(z)\right| = \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}|z-1|^2\right).$$

□

We now consider what happens when z is not a complex number but a unitary or unitary path.

Corollary 5.6. *Let* $u \in U(A)$ *with* $||u - 1|| < 2$. *Then* $a := \frac{1}{2\pi i}\ln(u)$, *defined by functional calculus, satisfies*

- a *is selfadjoint.*
- $\|a\| = \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}\|u-1\|^2\right)$.
- $u = e^{2\pi ia}$.

In particular for some $t_0, t_1 \in [0,1]$ with $t_0 < t_1$, if $u : [t_0, t_1] \to U(A)$ is continuous/piecewise smooth, and $\sup_{t\in[t_0,t_1]} \|u(t) - 1\| < 2$ then $a(t) : [t_0, t_1] \to A_{sa}$ defined by $a(t) = \frac{1}{2\pi i}\ln(u(t))$ satisfies

- *$a(t)$ is continuous/piecewise smooth,*
- *$\sup_{t\in[t_0,t_1]} \|a(t)\| = \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}\sup_{t\in[t_0,t_1]} \|u(t) - 1\|^2\right)$,*
- *$u(t) = e^{2\pi i a(t)}$.*

Proof. First note for all $t, s \in [0,1]$, $\sigma(u(t)u^*(s)) \subseteq \mathbb{T}\backslash\{-1\}$. Indeed suppose $-1 \in \sigma(u(t)u^*(s))$, then

$$||u(t)u^*(s) - 1|| \geq |-1-1| = 2.$$

Hence we can make the complex logarithm is continuous by taking a branch cut at -1 and hence a and $a(t)$ are well defined by functional calculus. We show a is self-adjoint. Since a is normal with spectrum

$$\begin{aligned}\sigma\left(\frac{1}{2\pi i}\ln(u)\right) &= \left\{\frac{1}{2\pi i}\ln(z) \mid z \in \sigma(u)\right\} \\ &\subseteq \left\{\frac{1}{2\pi i}\ln(z) \mid z \in \mathbb{T}\backslash\{-1\}\right\} \\ &= \left(-\frac{1}{2}, \frac{1}{2}\right).\end{aligned}$$

a is selfadjoint. We compute the norm of a. By Lemma 5.5

$$\begin{aligned}||a|| &= \sup_{z\in\sigma(u)} \left|\frac{1}{2\pi i}\ln(z)\right| \\ &= \sup_{z\in\sigma(u)} \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}|z-1|^2\right).\end{aligned}$$

arccos is a decreasing function so $\arccos\left(1 - \frac{1}{2}|z-1|^2\right)$ is large when $1 - \frac{1}{2}|z-1|^2$ is small, thus

$$\begin{aligned}\sup_{z\in\sigma(u)} \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}|z-1|^2\right) &= \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}\sup_{z\in\sigma(u)}|z-1|^2\right) \\ &= \frac{1}{2\pi}\arccos\left(1 - \frac{1}{2}\|u-1\|^2\right).\end{aligned}$$

The properties for $u : [t_0, t_1] \to U(A)$ being a continuous/piecewise smooth path follows from analogous computations. □

Having seen that we can write unitaries and unitary paths in exponential form, we are ready to prove a lemma, which will by paramount to proving the desired properties of the de la Harpe-Skandalis determinant map.

Lemma 5.7. *Let A be a unital C^*-algebra and let $u : [0,1] \to U_n(A)$ be a continuous path of unitaries. Then for every $0 < \varepsilon < 2$ there exists a piecewise smooth path $v : [0,1] \to U_n(A)$ such that $u(0) = v(0)$, $u(1) = v(1)$ and*

$$\sup_{t\in[0,1]} ||u(t) - v(t)|| < \varepsilon.$$

In particular u is homotopic to v, $u \sim_h v$.

Proof. Since u is continuous we find $0 = t_0 < t_1 < ... < t_k = 1$ such that

$$\sup_{t\in[t_{j-1},t_j]} ||u(t)u^*(t_{j-1}) - 1|| = \sup_{t\in[t_{j-1},t_j]} ||u(t) - u(t_{j-1})|| < \frac{\varepsilon}{2}.$$

For $j \in \{1, 2, ..., k\}$ define $a_j : [t_{j-1}, t_j] \to M_n(A)_{sa}$ by functional calculus

$$a_j(t) = \frac{1}{2\pi i} \ln(u(t)u^*(t_{j-1})),$$

thus for $t \in [t_{j-1}, t_j]$, $u(t) = e^{2\pi i a_j(t)} u(t_{j-1})$.

By Corollary 5.6 a_j is well defined and

$$\sup_{t\in[t_{j-1},t_j]} ||a_j(t)|| = \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2} \sup_{t\in[t_0,t_1]} \|u(t) - 1\|^2\right) < \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\left|\frac{\varepsilon}{2} - 1\right|^2\right).$$

Define $\widetilde{a}_j : [t_{j-1}, t_j] \to M_n(A)_{sa}$ by

$$\widetilde{a}_j(t) = \frac{t - t_{j-1}}{t_j - t_{j-1}} a_j(t_j).$$

Note $\widetilde{a}_j$ is smooth and

$$\sup_{t\in[t_{j-1},t_j]} ||\widetilde{a}_j(t)|| = ||a_j(t_j)|| < \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\left|\frac{\varepsilon}{2} - 1\right|^2\right),$$

with the last inequality coming from the bound above. Using $\widetilde{a}_j$ we define a continuous path by

$$v(t) = \begin{cases} e^{2\pi i \widetilde{a}_1(t)} u(0) \text{ if } t \in [0, t_1], \\ \quad \vdots \\ e^{2\pi i \widetilde{a}_k(t)} u(t_{k-1}) \text{ if } t \in [t_{k-1}, 1]. \end{cases}$$

First note

$$\begin{aligned} v(0) &= e^{2\pi i \widetilde{a}_1(0)} u(0) = u(0), \\ v(1) &= e^{2\pi i \widetilde{a}_k(1)} u(t_{k-1}) = e^{2\pi i a_k(1)} u(t_{k-1}) = u(1)u(t_{k-1})^* u(t_{k-1}) = u(1). \end{aligned}$$

Moreover for each j, $e^{2\pi i \widetilde{a}_j(t_j)} u(t_{j-1})$ is smooth hence v will be piecewise smooth if the endpoints are connected. For $j \in \{1, 2, ..., k-1\}$ we compute

$$\begin{aligned} e^{2\pi i \widetilde{a}_j(t_j)} u(t_{j-1}) &= e^{2\pi i a_j(t_j)} u(t_{j-1}) = u(t_j) \\ e^{2\pi i \widetilde{a}_{j+1}(t_j)} u(t_j) &= e^0 u(t_j) = u(t_j). \end{aligned}$$

We show $||u - v|| < \varepsilon$. It is enough to check $\sup_{t\in[t_{j-1},t_j]} ||u(t) - v(t)|| < \varepsilon$. Using our bounds $||a_j(t)|| < \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\left|\frac{\varepsilon}{2} - 1\right|^2\right)$ and $\|\widetilde{a}_j(t)|| < \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\left|\frac{\varepsilon}{2} - 1\right|^2\right)$, so by functional calculus

$$\begin{aligned} \sup_{t\in[t_{j-1},t_j]} ||e^{2\pi i a_j(t)} - 1|| &< 1 - \frac{\varepsilon}{2}, \\ \sup_{t\in[t_{j-1},t_j]} ||e^{2\pi i \widetilde{a}_j(t)} - 1|| &< 1 - \frac{\varepsilon}{2}. \end{aligned}$$

Thus by the triangle inequality

$$\sup_{t\in[t_{j-1},t_j]} ||u(t)-v(t)|| < \varepsilon.$$

Lastly u is homotopic to v by Proposition 2.1.3 in [RLL00]. □

We are now able to prove the main theorem of this section.

Proposition 5.8 ([CGS+23][2.11]). *Let A be a unital C^*-algebra. The de la Harpe-Skandalis determinant map gives a continuous group homomorphism*

$$\widetilde{\Delta}_A : U_\infty(C([0,1],A)) \to \mathrm{Aff}T(A)$$

such that

(i) $\widetilde{\Delta}_A(u)$ depends only on the homotopy class of u, in the space of continuous unitary paths that fixes endpoints.

(ii) For a self-adjoint $h \in M_n(C([0,1],A))$,

$$\widetilde{\Delta}_A(e^{2\pi ih})(\tau) = \tau_n(h(1)-h(0)), \quad \tau \in T(A),$$

where τ_n is the canonical non-normalized extension of τ to $M_n(A)$.

(iii) Let $(C_0((0,1),A))^\dagger$ denote the unitisation of $C_0((0,1),A)$, and identify the homotopy classes of $U_\infty(C_0((0,1),A))^\dagger)$ with $K_0(A)$ by applying the Bott map. Then the homomorphism from (i) is given by the pairing map ρ_A.

(iv) There is a continuous group homomorphism

$$\det_A : U_\infty^{(0)}(A) \to \mathrm{Aff}T(A)/\overline{\mathrm{Im}\rho_A},$$

given by $\det_A(u) := \widetilde{\Delta}_A(v) + \overline{\mathrm{Im}\rho_A}$, where $v \in U_\infty(C([0,1],A))$ has $v(0)=1_A$ and $v(1)=u$.

Proof. We start by showing the de la Harpe-Skandalis determinant map is a group homomorphism. Let $u,v : [0,1] \to U_n(A)$ be piecewise smooth paths of unitaries, and $\tau \in T(A)$ be a trace. Then

$$\begin{aligned}
2\pi i\widetilde{\Delta}_A(uv)(\tau) &= \int_0^1 \tau_n\left(\left(\frac{d}{dt}u(t)v(t)\right)v^*(t)u^*(t)\right)dt \\
&= \int_0^1 \tau_n\left(\left(\frac{d}{dt}u(t)\right)v(t)v^*(t)u^*(t) + u(t)\left(\frac{d}{dt}v(t)\right)v^*(t)u^*(t)\right)dt \\
&= 2\pi i\widetilde{\Delta}_A(u)(\tau) + \int_0^1 \tau_n\left(v^*(t)u^*(t)u(t)\left(\frac{d}{dt}v(t)\right)\right)dt \\
&= 2\pi i(\widetilde{\Delta}_A(u) + \widetilde{\Delta}_A(v)).
\end{aligned}$$

We prove (i). Let $u,v : [0,1] \to U_n(A)$ be piecewise smooth paths of unitaries such that $u(0)=v(0)$, $u(1)=v(1)$, and let $\|u-v\| < 2$. Define $a : [0,1] \to M_n(A)_{sa}$ by

$$a(t) = \frac{1}{2\pi i}\ln(u(t)^*v(t)).$$

By Corollary 5.6 a is well defined and piecewise smooth, moreover as u and v have the same endpoints

$$a(0) = 0 = a(1).$$

Computing the determinant

$$\begin{aligned}
\widetilde{\Delta}_A(u^*v)(\tau) &= \widetilde{\Delta}_A(e^{2\pi i a})(\tau) \\
&= \frac{1}{2\pi i}\int_0^1 \tau_n\left(\left(\frac{d}{dt}e^{2\pi i a(t)}\right)e^{-2\pi i a(t)}\right)dt \\
&= \frac{1}{2\pi i}\int_0^1 \tau_n\left(2\pi i\left(\frac{d}{dt}a(t)\right)e^{2\pi i a(t)}e^{-2\pi i a(t)}\right)dt \\
&= \tau_n\left(\int_0^1 \frac{d}{dt}a(t)dt\right) \\
&= \tau_n(a(1)-a(0)) \\
&= 0.
\end{aligned}$$

Thus $\widetilde{\Delta}_A(u) = \widetilde{\Delta}_A(v)$ since $\widetilde{\Delta}_A$ is a group homomorphism.
Now assume that $u_1, u_2 : [0,1] \to U_n(A)$ are piecewise smooth paths of unitaries, and $u_1 \sim_h u_2$. As $u_1 \sim_h u_2$ there exists continuous paths $v_1, v_2, ..., v_k : [0,1] \to U_n(A)$ such that $u_1(0) = u_2(0) = v_j(0)$, $u_1(1) = u_2(1) = v_j(1)$, for $j \in \{1,2,...,k\}$, further $\|u_1 - v_1\| < 1$, $\|u_2 - v_k\| < 1$ and $\|v_{j-1} - v_j\| < 1$. By Lemma 5.7 find piecewise smooth paths $w_j : [0,1] \to U_n(A)$ such that $w_j(0) = v_j(0)$, $w_j(1) = v_j(1)$ and $\|w_j - v_j\| < \frac{1}{2}$. Applying the triangle inequality $\|u_1 - w_1\| < 2$, $\|u_2 - w_k\| < 2$ and $\|w_{j-1} - w_j\| < 2$, thus

$$\widetilde{\Delta}(u_1) = \widetilde{\Delta}(w_1) = \widetilde{\Delta}(w_2) = ... = \widetilde{\Delta}(w_k) = \widetilde{\Delta}(u_2),$$

by the first part. So we have proven (i) for piecewise smooth paths. We extend the de la Harpe-Skandalis determinant map to continuous paths of unitaries. Let $u : [0,1] \to U_n(A)$ be a continuous path of unitaries, and let $u' : [0,1] \to U_n(A)$ be a piecewise smooth path homotopic to u, note at least 1 exists by Lemma 5.7, define the extension by $\widetilde{\Delta}_A(u) := \widetilde{\Delta}_A(u')$. This is well defined by the above, indeed if we have two piecewise smooth paths, $u_1, u_2 : [0,1] \to U_n(A)$ both homotopic to u, then u_1 is homotopic to u_2, thus $\widetilde{\Delta}_A(u_1) = \widetilde{\Delta}_A(u_2)$. One observes $\widetilde{\Delta}_A$ is still a group homomorphism by an analogous computation. To see that the extension only depends on the homotopy class of u, let $v : [0,1] \to U_n(A)$ be a continuous path of unitaries homotopic to u, then for some piecewise smooth path of unitaries $v' : [0,1] \to U_n(A)$ homotopic to v,

$$u' \sim_h u \sim_h v \sim_h v',$$

and by the above

$$\widetilde{\Delta}_A(u) = \widetilde{\Delta}_A(u') = \widetilde{\Delta}_A(v') = \widetilde{\Delta}_A(v).$$

We prove (ii). Let $h : [0,1] \to M_n(A)$ be a selfadjoint piecewise smooth map,

$$\begin{aligned}
\widetilde{\Delta}_A(e^{2\pi i h})(\tau) = \frac{1}{2\pi i}\int_0^1 \tau_n\left(\left(\frac{d}{dt}e^{2\pi i h(t)}\right)e^{-2\pi i h(t)}\right)dt &= \int_0^1 \tau_n\left(\left(\frac{d}{dt}h(t)\right)\right)dt \\
&= \tau_n(h(1)-h(0)).
\end{aligned}$$

For $h \in M_n(C([0,1],A))$, find a piecewise smooth path homotopic to h with the same endpoints using Lemma 5.7, applying by property (i) then yields property (ii).

We show $\widetilde{\Delta}_A$ is continuous. Let $\varepsilon > 0$, $\delta = \sqrt{2(1 - \cos\left(\frac{\pi}{n}\varepsilon\right))}$, and $u, v : [0,1] \to U_n(A)$ be continuous paths of unitaries, if

$$\sup_{t\in[0,1]} \|u(t)v(t)^* - 1\| = \sup_{t\in[0,1]} \|u(t) - v(t)\| < \delta,$$

define $a : [0,1] \to M_n(A)_{sa}$ by $a(t) = \frac{1}{2\pi i} \ln(u(t)v(t)^*)$. By Corollary 5.6 a is well defined and

$$\|a(t)\| = \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\|u(t)v(t)^* - 1\|^2\right) < \frac{1}{2\pi} \arccos\left(1 - \frac{1}{2}\delta^2\right) = \frac{1}{2n}\varepsilon.$$

For $\tau \in T(A)$ and by property (ii),

$$\begin{aligned}
|(\widetilde{\Delta}_A(u) - \widetilde{\Delta}_A(v))(\tau)| &= |\widetilde{\Delta}_A(uv^*)(\tau)| \\
&= |\widetilde{\Delta}_A(e^{2\pi i a})| \\
&= |\tau_n(a(1)) - \tau_n(a(0))| \\
&\leq |\tau_n(a(1))| + |\tau_n(a(0))| \\
&\leq n(\|a(1)\| + \|a(0)\|) \\
&< n\left(\frac{1}{2n}\varepsilon + \frac{1}{2n}\varepsilon\right) \\
&= \varepsilon.
\end{aligned}$$

So we have continuity for all $n \in \mathbb{N}$ and thus the de la Harpe-Skandalis map is continuous on $U_\infty(C([0,1], A))$. We show property (iii). Recall the Bott map for the cone-suspension exact sequence, β_{SA}, associates a projection, p, to the smooth unitary path $f_p : [0,1] \to U_n(A)$ given by $f_p(t) = e^{2\pi i t p}$. Thus $\beta_{SA}([p]_0 - [q]_0) = e^{2\pi i t p} e^{-2\pi i t q}$. By property (ii)

$$\widetilde{\Delta}_A(\beta_{SA}([p]_0 - [q]_0)) = \widetilde{\Delta}_A(e^{2\pi i t p}) - \widetilde{\Delta}_A(e^{2\pi i t q}) = \widehat{p} - \widehat{q} = \rho_A([p]_0 - [q]_0).$$

We show property (iv). Observe $U_\infty^{(0)}(A) = \bigcup_{n\in\mathbb{N}} U_n^{(0)}(A)$, thus for $u \in U_\infty^{(0)}(A)$ there exists some $n \in \mathbb{N}$ such that $u \in U_n^{(0)}(A)$, further as u is connected to the identity, find a path $v \in U_n^{(0)}(C([0,1], A))$ such that $v(0) = 1_{M_n(A)}$ and $v(1) = u$. To check that $\det_A$ is well defined we show two different paths from $1_{M_n(A)}$ to u yields the same element modulo $\mathrm{Im}(\rho_A)$. Let $v_1, v_2 : [0,1] \to U_n(A)$ be continuous paths of unitaries, such that $v_1(0) = v_2(0) = 1_{M_n(A)}$ and $v_1(1) = v_2(1) = u$. Then

$$v_1(0)v_2^*(0) = 1_{M_n(A)} = uu^* = v_1(1)v_2^*(1),$$

so $v_1 v_2^* \in U_n(C_0((0,1), A)^\dagger)$, and by part (iii), $\widetilde{\Delta}_A(v_1 v_2^*) \in \mathrm{Im}(\rho_A)$. Since $\widetilde{\Delta}_A$ is a continuous group homomorphism

$$0 + \overline{\mathrm{Im}(\rho_A)} = \widetilde{\Delta}_A(v_1) - \widetilde{\Delta}_A(v_2) + \overline{\mathrm{Im}(\rho_A)}$$

or equivalently

$$\widetilde{\Delta}_A(v_1) + \overline{\mathrm{Im}(\rho_A)} = \widetilde{\Delta}_A(v_2) + \overline{\mathrm{Im}(\rho_A)}.$$

Note $\det_A$ is a group homomorphism as $\widetilde{\Delta}_A$ is so what is left to show is continuity. As $U_n^{(0)}$ is a topological group it is enough to show continuity at 1, for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $\varepsilon > 0$, $\delta = \sqrt{2\left(1 - \cos\left(\frac{2\pi}{n}\varepsilon\right)\right)}$, and $u \in U_n^{(0)}(A)$, if

$$\|u - 1\| < \delta,$$

define $a = \frac{1}{2\pi i}\ln(u)$, by Corollary 5.6 a is well defined and

$$\|a\| = \frac{1}{2\pi}\arccos\left(1-\frac{1}{2}\|u-1\|^2\right) < \frac{1}{2\pi}\arccos\left(1-\frac{1}{2}\delta^2\right) = \frac{1}{n}\varepsilon.$$

Define the smooth path $b : [0,1] \to M_n(A)_{sa}$ by $b(t) = ta$, which is self-adjoint as a is, and define $v : [0,1] \to U_n(A)$ by $v(t) = e^{2\pi i b(t)}$ which is a smooth path of unitaries, note $v(0) = 1$ and $v(t) = u$. By property (ii)

$$\widetilde{\Delta}_A(v) = \widehat{b(1)} - \widehat{b(0)} = \widehat{a}.$$

For $\tau \in T(A)$,

$$|(\det_A(u) - \det_A(1))(\tau)| = |\widetilde{\Delta}_A(v)(\tau)| = |\tau_n(a)| \leq n\|a\| < \varepsilon.$$

$\square$

Being able to work with the de la Harpe-Skandalis determinant map will enable us to construct a short exact sequence using the Thomsen and pairing maps.

Properties of the Thomsen map

We are now back to the Thomsen map, and in this section we construct the short exact sequence alluded to earlier.

Proposition 5.9 ([CGS+23](2.9)). *Let A be a unital C^*-algebra then*

(i) $\mathrm{Im}(\mathrm{Th}_A) = \ker(\not{a}_A)$,

(ii) $\ker(\mathrm{Th}_A) = \overline{\mathrm{Im}(\rho_A)}$.

Proof. We prove (i). Let $[x]_{\mathrm{alg}} \in \ker(\not{a}_A)$, then $x \in U_\infty^{(0)}(A)$, and by Proposition 2.1.6 in [RLL00]

$$x = e^{ih_1}e^{ih_2}...e^{ih_k},$$

for some $k \in \mathbb{N}$ and $h_j \in A_{sa},\ j \in \{1,...,k\}$. Denote by $l_j = \frac{1}{2\pi}h_j$ which is also self-adjoint. Computing Th_A

$$\mathrm{Th}_A(\widehat{l_1}\widehat{l_2}\cdot ... \cdot \widehat{l_k}) = [e^{2\pi i l_1 l_2 \cdot ... \cdot l_k}]_{\mathrm{alg}} = [e^{ih_1}e^{ih_k}...e^{ih_k}]_{\mathrm{alg}} = [x]_{\mathrm{alg}}.$$

and hence $\ker(\not{a}_A) \subseteq \mathrm{Im}(\mathrm{Th}_A)$. We show the converse inclusion. By Lemma 2.1.3 in [RLL00], $e^{2\pi i a}$ is connected to the identity for all self-adjoint elements $a \in A_{sa}$ and hence $\not{a}_A(\mathrm{Th}_A(\widehat{a})) = 0$, thus $\mathrm{Im}(\mathrm{Th}_A) \subseteq \ker(\psi_A)$.

We prove (ii). Note if $u \in DU_\infty^{(0)}(A)$ then $u = aba^*b^*$ for some $a, b \in U_\infty^{(0)}(A)$, and as $\widetilde{\Delta}_A$ is a homomorphism

$$\det_A(u) = \widetilde{\Delta}_A(v_a v_b v_{a^*} v_{b^*}) + \overline{\mathrm{Im}(\rho_A)} = \widetilde{\Delta}_A(v_a) + \widetilde{\Delta}_A(v_b) + \widetilde{\Delta}_A(v_{a^*}) + \widetilde{\Delta}_A(v_{b^*}) + \overline{\mathrm{Im}(\rho_A)} = 0.$$

hence $DU_\infty(A) \subseteq \ker(\det_A)$, and by continuity of $\det_A$ we get $\overline{DU_\infty(A)} \subseteq \ker(\det_A)$. Let

$$\overline{\det}_A : U_\infty^{(0)}/\overline{DU_\infty^{(0)}(A)} \to \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)},$$

be the induced continuous group homomorphism. For $f \in \mathrm{Aff}T(A)$, $f = \widehat{a}$, $a \in A_{sa}$, and $\tau \in T(A)$

$$\overline{\det}_A(\mathrm{Th}_A(\widehat{a}))(\tau) = \overline{\det}_A(e^{2\pi i a})(\tau).$$

Consider the self-adjoint element $b \in C([0,1], A)$ given by $b(t) = ta$ then $v(t) = e^{2\pi i b(t)}$ is a continuous path of unitaries such that $v(0) = 1_A$ and $v(1) = e^{2\pi i a}$. Continuing our calculation and using property (ii) from Proposition 5.8

$$\overline{\det}_A(e^{2\pi i a})(\tau) = \widetilde{\Delta}_A(v)(\tau) + \overline{\mathrm{Im}(\rho_A)} = \tau(b(1) - b(0)) + \overline{\mathrm{Im}(\rho_A)} = \tau(a) + \overline{\mathrm{Im}(\rho_A)} = \widehat{a}(\tau) + \overline{\mathrm{Im}(\rho_A)}.$$

Thus when $\widehat{a} \in \ker(\mathrm{Th}_A)$ then $\mathrm{Th}_A(\widehat{a}) \in \ker(\overline{\det}_A)$ which implies $\mathrm{Th}_A(\widehat{a}) \in \overline{\mathrm{Im}(\rho_A)}$, hence $\ker(\mathrm{Th}_A) \subseteq \overline{\mathrm{Im}(\rho_A)}$. We show $\overline{\mathrm{Im}(\rho_A)} \subseteq \ker(\mathrm{Th}_A)$. Let $p, q \in \mathcal{P}_n(A)$ be projections, then

$$\mathrm{Th}_A(\rho_A([p]_0 - [q]_0)) = \mathrm{Th}_A(\widehat{p}_n - \widehat{q}_n) = [e^{2\pi i p} e^{-2\pi i q}]_{\mathrm{alg}} = [e^{2\pi i p}]_{\mathrm{alg}} + [e^{-2\pi i q}]_{\mathrm{alg}} = 0,$$

where the last equality comes from the spectrum of a projection is contained in $\{0, 1\}$ and we have defined $e^{2\pi i p}$ using functional calculus. Thus $\mathrm{Im}(\rho_A) \subseteq \ker(\mathrm{Th}_A)$. Recall by Proposition 3.21 for $f \in \mathrm{Aff}T(A)$ we can find a $a \in A_{sa}$ such that $\widehat{a} = f$, and $\|a\| < \|f\| + \varepsilon$, hence the Thomsen map is continuous into $U_1(A)/\overline{DU_1(A)}$ and $\overline{\mathrm{Im}(\rho_A)} \subseteq \ker(\mathrm{Th}_A)$ finishing the proposition. □

We are finally ready to reap the reward of the work in this section, and construct the short exact sequence.

Corollary 5.10 ([CGS+23]). *Let A be a unital C^*-algebra, Th_A it's associated Thomsen map and $\not{a}_A$ be the canonical surjection from $\overline{K}_1^{\mathrm{alg}}(A) \to K_1(A)$. Then*

$$0 \longrightarrow \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \xrightarrow{\overline{\mathrm{Th}}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not{a}_A} K_1(A) \longrightarrow 0.$$

is a short exact sequence.

Proof. Using Noether first isomorphism theorem

$$\overline{\mathrm{Th}}_A : \mathrm{Aff}T(A)/\ker(\mathrm{Th}_A) \to \mathrm{Im}(\mathrm{Th}_A),$$

is a group isomorphism. Then applying $(i), (ii)$ from Proposition 5.9 we get

$$\overline{\mathrm{Th}}_A : \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \to \ker(\not{a}_A).$$

To construct the sequence we compose $\overline{\mathrm{Th}_A}$ with the inclusion $\iota : \ker(\not{a}_A) \to \overline{K}_1^{\mathrm{alg}}(A)$. We will abuse notation and write $\overline{\mathrm{Th}_A}$ even when we have composed with the inclusion, thus

$$0 \longrightarrow \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \xrightarrow{\overline{\mathrm{Th}}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not{a}_A} K_1(A) \longrightarrow 0,$$

is short exact. □

This sequence actually splits, but to see this we first have to introduce injective modules and divisible abelian groups. Our introduction will be very short, and many of the proofs will be omitted.

Definition 5.11. Let R be a ring, then a left module M over R is injective if it satisfies one of the following equivalent conditions

- If M is a submodule of another left R module, N, then there exists another submodule of N, K, such that the internal direct sum $M + K = N$ and $M \cap K = \{0\}$.
- Any short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} K \longrightarrow 0,$$

  of left R modules splits
- If X and Y are left R modules, $f : X \to Y$ is an injective homomorphism and $g : X \to M$ is any homomorphism, then there exists $h : Y \to M$ such that the diagram with exact rows,

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y \\ & & g\big\downarrow & \swarrow_{h} & \\ & & M & & \end{array}$$

  commutes.

We define divisible abelian groups

Definition 5.12 ([CGS+23]). Let A be an abelian group. We say A is divisible if for each $a \in A$ and $n \in \mathbb{N}$ there exists $b \in A$ such that $nb = a$.

There is a correspondence between divisible abelian groups and injective modules.

Proposition 5.13. *Let A be an abelian group then A is divisible if and only if A is an injective module.*

A neat result about divisible groups is that divisibility is preserved under quotients.

Proposition 5.14. *Let A be a divisible group and $B \subset A$ be a normal subgroup, then A/B is divisible.*

Proof. Let $n \in \mathbb{N}$ and $[a]_B \in A/B$. Using the canonical surjection $\pi : A \to A/B$, $\pi(a) = [a]_B$, and since A is divisible there exists $a' \in A$ such that $na' = a$. Hence

$$[a]_B = \pi(a) = n\pi(a') = n[a']_B.$$

□

We show that any real vector space is a divisible group, this would show $\mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)}$ is divisible.

Proposition 5.15. *Let V be a real vector space, then V is a divisible group.*

Proof. Let $v \in V$ and $n \in \mathbb{N}$. Since V is a real valued vector space $w = \frac{1}{n}v$ is a well defined element and hence $ng = f$. □

Using the above we see our sequence splits.

Corollary 5.16 ([CGS+23] (2.9)). *Let A be a unital C^*-algebra, then the short exact sequence*

$$0 \longrightarrow \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \xrightarrow{\overline{\mathrm{Th}}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not{a}_A} K_1(A) \longrightarrow 0.$$

from Corollary 5.10 splits. Thus $\overline{K}_1^{\mathrm{alg}}(A) \cong \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \oplus K_1(A)$.

Proof. By Proposition 5.15, $\mathrm{Aff}T(A)$ is divisible and by Proposition 5.14 $\mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)}$ is divisible, hence injective by Proposition 5.13. By the definition of injective modules 5.11

$$0 \longrightarrow \mathrm{Aff}T(A)/\overline{\mathrm{Im}(\rho_A)} \xrightarrow{\overline{\mathrm{Th}}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{a\!\!\!/_A} K_1(A) \longrightarrow 0.$$

splits. □

We are now only missing one ingredient in the total invariant, which is a family of homomorphism.

6 The total invariant

In this section we will define the last ingredient missing from the total invariant, namely a collection of maps, $\zeta_A^{(n)}$, connecting K-theory with coefficients and $\overline{K}_1^{\mathrm{alg}}$. Then we finally define the total invariant for a unital C^*-algebra.

6.1 The zeta maps

In this section we will introduce a collection of maps, $\zeta_A^{(n)}$, and show how these maps arises in a natural fashion. Let A be a unital C^*-algebra and for now fix $n \geq 2$. Recall we can identify the unitisation of $\mathbb{I}_n(A)$ by

$$\mathbb{I}_n(A)^\dagger = \{f \in C([0,1], M_n(A)) \mid f(0) \in A \otimes 1_{M_n},\ f(1) \in \mathbb{C}1_{M_n(A)}\}.$$

Consider the evaluation maps, $\mathrm{ev}_A^{(0,n)} : \mathbb{I}_n(A)^\dagger \to A$ and $\mathrm{ev}_A^{(1,n)} : \mathbb{I}_n(A)^\dagger \to A$. Where the former is the map induced by evaluation at 0, be aware that the co-domain of $\mathrm{ev}_A^{(0,n)}$ is A and not $A \otimes 1_{M_n}$ thus $\mathrm{ev}_A^{(0,n)}(1_{(\mathbb{I}_n(A)^\dagger)}) = 1_A$. The latter map is induced by evaluation at 1, again we caution the reader to be aware of the co-domain. Let $\widetilde{\zeta}_A^{(n)} : U_\infty(\mathbb{I}_n(A)^\dagger) \to \overline{K}_1^{\mathrm{alg}}(A)$ be given by

$$\widetilde{\zeta}_A^{(n)}(u) := [\mathrm{ev}_A^{(0,n)}(u)]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,n)}(u)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right),$$

where the $\frac{1}{n}$ term is used as we have earlier chosen our trace extensions to be non-normalised and the evaluation maps are normalised. Note $\widetilde{\zeta}_A^{(n)}$ is well defined as each constituent is well defined on $\overline{K}_1^{\mathrm{alg}}(A)$. By Proposition 5.9 and the definition of $\mathrm{ev}_A^{(1,n)}$, the canonical surjection $a\!\!\!/_A : \overline{K}_1^{\mathrm{alg}} \to K_1(A)$ applied to $\widetilde{\zeta}_A^{(n)}$ kills the last two terms, thus

$$a\!\!\!/_A(\widetilde{\zeta}_A^{(n)}(u)) = \left[\mathrm{ev}_A^{(0,n)}\right]_1.$$

Recall $K_0(A;\mathbb{Z}_n) := K_1(\mathbb{I}_n(A))$ and the Bockstein operation $\nu_{0,A}^{(n)} : K_0(A;\mathbb{Z}_n) \to K_1(A)$ was defined as the induced map $K_1\left(\mathrm{ev}_A^{(0,n)}\right)$ therefore

$$a\!\!\!/_A(\widetilde{\zeta}_A^{(n)}(u)) = \nu_{0,A}^{(n)}([u]_1).$$

The reason we defined $\widetilde{\zeta}_A^{(n)}$ is they induce the group homomorphisms $\zeta_A^{(n)}$ we want.

Proposition 6.1 ([CGS+23](3.1))**.** *Let A be unital C^*-algebra and let $n \geq 2$. Then the map $\widetilde{\zeta}_A^{(n)} : U_\infty(\mathbb{I}_n(A)^\dagger) \to \overline{K}_1^{\mathrm{alg}}(A)$ induce a group homomorphism $\zeta_A^{(n)} : K_0(A;\mathbb{Z}_n) \to \overline{K}_1^{\mathrm{alg}}(A)$. Further this is natural in A and $\nu_{0,A}^{(n)} = a\!\!\!/_A \circ \zeta_A^{(n)}$.*

Proof. We show $\widetilde{\zeta}_A^{(n)}$ is a group homomorphism. Let $u, v \in U_\infty(\mathbb{I}_n(A)^\dagger)$, and recall both $\widetilde{\Delta}_A$ and Th_A are group homomorphisms,

$$\begin{aligned}\widetilde{\zeta}_A^{(n)}(uv) &= [\mathrm{ev}_A^{(0,n)}(uv)]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,n)}(uv)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(uv)\right) \\ &= [u(0)v(0)]_{\mathrm{alg}} - [u(1)v(1)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right) + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(v)\right) \\ &= \widetilde{\zeta}_A^{(n)}(u) + \widetilde{\zeta}_A^{(n)}(v).\end{aligned}$$

We show $\widetilde{\zeta}_A^{(n)}$ induces a group homomorphism on $K_0(A;\mathbb{Z}_n)$ hence we show $\widetilde{\zeta}_A^{(n)}$ vanishes on $U_\infty^{(0)}(\mathbb{I}_n(A)^\dagger)$. By Proposition 2.1.6 in [RLL00] for any $u \in U_\infty^{(0)}(\mathbb{I}_n(A)^\dagger)$ we find $k, m \in \mathbb{N}$ and $a_1, a_2, ...a_m \in M_k(\mathbb{I}_n(A))_{sa}^\dagger$ such that

$$u = e^{ia_1}e^{ia_2}\cdot ... \cdot e^{ia_m}.$$

Since $\widetilde{\zeta}_A^{(n)}$ is a homomorphism it is enough to check that $\widetilde{\zeta}_A^{(n)}(e^{2\pi ia})$ vanishes for one $a \in M_k(\mathbb{I}_n(A))_{sa}^\dagger$. Let $\tau \in T(A)$ by Proposition 5.9 *(ii)*

$$\frac{1}{n}\widetilde{\Delta}_A(e^{2\pi ia})(\tau) = \frac{1}{n}\tau_{nk}(a(1) - a(0)) = \tau_k(\mathrm{ev}_A^{(1,n)}(a) - \mathrm{ev}_A^{(0,n)}(a)),$$

where the factor $\frac{1}{n}$ disappears by our normalisation conventions. The calculation implies $\frac{1}{n}\widetilde{\Delta}_A(e^{2\pi ia})$ and $(\mathrm{ev}_A^{(1,n)}(a) - \mathrm{ev}_A^{(0,n)})(a)$ induce the same affine function. Thus

$$\begin{aligned}\mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A\left(e^{2\pi ia}\right)\right) &= \left[e^{2\pi i(\mathrm{ev}_A^{(1,n)}(a) - \mathrm{ev}_A^{(0,n)}(a))}\right]_{\mathrm{alg}} \\ &= \left[e^{2\pi i\mathrm{ev}_A^{(1,n)}(a)}\right]_{\mathrm{alg}} - \left[e^{2\pi i\mathrm{ev}_A^{(0,n)}(a)}\right]_{\mathrm{alg}} \\ &= [\mathrm{ev}_A^{(1,n)}(e^{2\pi ia})]_{\mathrm{alg}} - [\mathrm{ev}_A^{(0,n)}(e^{2\pi ia})]_{\mathrm{alg}}.\end{aligned}$$

Hence

$$\widetilde{\zeta}_A^{(n)}(e^{2\pi ia}) = [\mathrm{ev}_A^{(0,n)}(e^{2\pi ia}))]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,n)}(e^{2\pi ia}))]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}(u)\right) = 0.$$

We show naturality of $\zeta_A^{(n)}$. Note it suffices to check naturality of each constituent, $\mathrm{ev}_A^{(0,n)}, \mathrm{ev}_A^{(1,n)}$, and $\mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}(u)\right)$. We show the evaluations are natural Let A, B be unital C^*-algebras, $\varphi : A \to B$ a unital *-homomorphism, and $i \in \{0, 1\}$, consider the diagram

$$\begin{array}{ccc} U_\infty(\mathbb{I}_n(A)^\dagger) & \xrightarrow{\mathrm{ev}_A^{(i,n)}} & \overline{K}_1^{\mathrm{alg}}(A) \\ {\scriptstyle \mathbb{I}_n(\varphi)}\downarrow & & \downarrow{\scriptstyle \overline{K}_1^{\mathrm{alg}}(\varphi)} \\ U_\infty(\mathbb{I}_n(B)^\dagger) & \xrightarrow[\mathrm{ev}_B^{(i,n)}]{} & \overline{K}_1^{\mathrm{alg}}(B). \end{array}$$

Let $u \in U_\infty(\mathbb{I}_n(A)^\dagger)$,

$$\overline{K}_1^{\mathrm{alg}}(\varphi)([\mathrm{ev}_A^{(i,n)}(u)]_{\mathrm{alg}}) = [\varphi(u(i))]_{\mathrm{alg}} = [\mathrm{ev}_B^{(i,n)}(\varphi \circ u)]_{\mathrm{alg}} = [\mathrm{ev}_B^{(i,n)}(\mathbb{I}_n(\varphi)(u))]_{\mathrm{alg}}.$$

We show naturality of $\mathrm{Th}_{-}\left(\frac{1}{n}\widetilde{\Delta}_{-}\right)$. Consider the diagram

$$\begin{array}{ccccc}
U_\infty(\mathbb{I}_n(A)^\dagger) & \xrightarrow{\frac{1}{n}\widetilde{\Delta}_A} & \mathrm{Aff}\,T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) \\
{\scriptstyle \mathbb{I}_n(\varphi)}\big\downarrow & & {\scriptstyle \mathrm{Aff}(T(\varphi))}\big\downarrow & & \big\downarrow{\scriptstyle \overline{K}_1^{\mathrm{alg}}(\varphi)} \\
U_\infty(\mathbb{I}_n(B)^\dagger) & \xrightarrow[\frac{1}{n}\widetilde{\Delta}_B]{} & \mathrm{Aff}\,T(B) & \xrightarrow[\mathrm{Th}_B]{} & \overline{K}_1^{\mathrm{alg}}(B).
\end{array}$$

We show the left square commutes. For $\tau \in T(B)$ we get $\mathrm{Aff}(T(\varphi))(f)(\tau) = f(\tau \circ \varphi)$, then as φ is a *-homomorphism

$$\begin{aligned}
\mathrm{Aff}(T(\varphi))\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right)(\tau) &= \frac{1}{n}\widetilde{\Delta}_A(u)(\tau \circ \varphi) \\
&= \frac{1}{2n\pi i}\int_0^1 (\tau \circ \varphi)_n \left(\left(\frac{d}{dt}u(t)\right) u^*(t)\right) dt \\
&= \frac{1}{2n\pi i}\int_0^1 \tau_n \left(\varphi\left(\frac{d}{dt}u(t)\right) \varphi(u^*(t))\right) dt \\
&= \frac{1}{2n\pi i}\int_0^1 \tau_n \left(\left(\frac{d}{dt}\varphi(u(t))\right) \varphi(u(t))^*\right) dt \\
&= \frac{1}{n}\widetilde{\Delta}_B(\varphi(u))(\tau).
\end{aligned}$$

We show the right square commutes. By Proposition 3.21 we can find $a \in A_{sa}$ such that $f = \widehat{a}$, then $\mathrm{Aff}(T(\varphi))(f)(\tau) = \widehat{a}(\tau \circ \varphi) = (\tau \circ \varphi)(a)$. Thus $\mathrm{Th}_B(\mathrm{Aff}(T(\varphi))(\widehat{a})) = [e^{2\pi i \varphi(a)}]_{\mathrm{alg}}$ and

$$[e^{2\pi i\varphi(a)}]_{\mathrm{alg}} = [\varphi(e^{2\pi i a})]_{\mathrm{alg}} = \overline{K}_1^{\mathrm{alg}}(\varphi)([e^{2\pi i a}]_{\mathrm{alg}}) = \overline{K}_1^{\mathrm{alg}}(\varphi)(\mathrm{Th}_A(\widehat{a})),$$

by functional calculus. Lastly $\not{a}_A(\zeta_A^{(n)}(u)) = \nu_{0,A}^{(n)}([u_1])$ by the same arguments as above. □

We now show how $\zeta_A^{(n)}$ fits into the other diagrams we have looked at so far.

Proposition 6.2 ([CGS$^+$23](3.2)). *Let A be a unital C^*-algebra and let $m, n \in \mathbb{N}$ and $m, n \geq 2$. Then*

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mu_{0,A}^{(n)}} & K_0(A;\mathbb{Z}_n) & \xrightarrow{\nu_{0,A}^{(n)}} & K_1(A) \\
{\scriptstyle \frac{1}{n}\rho_A}\big\downarrow & & {\scriptstyle \zeta_A^{(n)}}\big\downarrow & & \big\| \\
\mathrm{Aff}\,T(A) & \xrightarrow[\mathrm{Th}_A]{} & \overline{K}_1^{\mathrm{alg}}(A) & \xrightarrow[\not{a}_A]{} & K_1(A).
\end{array}$$

and

$$\begin{array}{ccccc}
K_0(A;\mathbb{Z}_{nm}) & \xrightarrow{\kappa_{0,A}^{(n,nm)}} & K_0(A;\mathbb{Z}_n) & \xrightarrow{\kappa_{0,A}^{(nm,n)}} & K_0(A;\mathbb{Z}_{nm}) \\
& {\scriptstyle m\zeta_A^{(nm)}}\searrow & {\scriptstyle \zeta_A^{(n)}}\big\downarrow & \swarrow{\scriptstyle \zeta_A^{(nm)}} & \\
& & \overline{K}_1^{\mathrm{alg}}(A) & &
\end{array}$$

commutes.

Proof. We show

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mu_{0,A}^{(n)}} & K_0(A;\mathbb{Z}_n) & \xrightarrow{\nu_{0,A}^{(n)}} & K_1(A) \\ {\scriptstyle \frac{1}{n}\rho_A}\downarrow & & {\scriptstyle \zeta_A^{(n)}}\downarrow & & \parallel \\ \mathrm{Aff}T(A) & \xrightarrow[\mathrm{Th}_A]{} & \overline{K}_1^{\mathrm{alg}}(A) & \xrightarrow[\not{a}_A]{} & K_1(A). \end{array}$$

commutes. We show the left square commutes. By Bott periodicity we may view $K_0(A)$ as $K_1(C_0((0,1),A))$, hence the Bockstein operation $\mu_{0,A}^{(n)}$ is the induced inclusion of $\iota : C_0((0,1),M_n(A)) \to \mathbb{I}_n(A)$ on K_1. So for $u \in U_\infty(C_0((0,1),A))^\dagger$, viewing u as a path, $u(0) = u(1) = U(\mathbb{C})$, and $\mu_{0,A}^{(n)}([u]_1) = [u]_{0,n} \in K_0(A;\mathbb{Z}_n)$, hence

$$\zeta_A^{(n)}(\mu_{0,A}^{(n)}([u]_1)) = \widetilde{\zeta}_A^{(n)}([u]_1).$$

By Proposition 5.9 *(iii)* $\widetilde{\Delta}_A(u) = \rho_A([u]_1)$, and since $\mathrm{ev}_A^{(0,n)}(u) = \mathrm{ev}_A^{(1,n)}(u)$,

$$\widetilde{\zeta}_A^{(n)}(u) = \mathrm{Th}_A\left(\frac{1}{n}\rho_A([u]_1)\right).$$

The right square commutes by Proposition 6.1. We show

$$\begin{array}{ccccc} K_0(A;\mathbb{Z}_{nm}) & \xrightarrow{\kappa_{0,A}^{(n,nm)}} & K_0(A;\mathbb{Z}_n) & \xrightarrow{\kappa_{0,A}^{(nm,n)}} & K_0(A;\mathbb{Z}_{nm}) \\ & {\scriptstyle m\zeta_A^{(nm)}}\searrow & {\scriptstyle \zeta_A^{(n)}}\downarrow & \swarrow{\scriptstyle \zeta_A^{(nm)}} & \\ & & \overline{K}_1^{\mathrm{alg}}(A) & & \end{array}$$

commutes. We start with the left triangle. For $f \in \mathbb{I}_n(M_m(A))$

$$f(0) = \mathcal{A} \otimes 1_{M_n}, \quad f(1) = (\alpha \otimes 1_{M_m}) \otimes 1_{M_n},$$

where $\mathcal{A} \in M_m(A)$, and $\alpha \in \mathbb{C}$ and for $g \in \mathbb{I}_{nm}(A)$

$$g(0) = a \otimes 1_{M_{nm}}, \quad g(1) = \beta \otimes 1_{M_{nm}},$$

where $a \in A$ and $\beta \in \mathbb{C}$. So $f(0) = g(0)$ if $\mathcal{A} = a \otimes 1_{M_m}$ and $f(1) = g(1)$ if $\alpha = \beta$. Recall that $\kappa_A^{(n,nm)}$ is the induced map of the inclusion, $\mathbb{I}_{nm}(A) \hookrightarrow \mathbb{I}_n(M_m(A))$, thus for $u \in U_k(\mathbb{I}_{nm}(A)^\dagger)$ $\kappa_A^{(n,nm)}([u]_1)$ satisfies that the endpoints are preserved. By our normalisation conventions

$$\begin{aligned} \zeta_A^{(n)}\left(\kappa_A^{(n,nm)}([u]_1)\right) &= [\mathrm{ev}_{M_m(A)}^{(0,n)}(u)]_{\mathrm{alg}} - [\mathrm{ev}_{M_m(A)}^{(1,n)}(u)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right) \\ &= [\mathrm{ev}_A^{(0,nm)}(u) \otimes 1_{M_m(A)}]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,nm)}(u) \otimes 1_{M_m(A)}]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right) \\ &= m\left([\mathrm{ev}_A^{(0,nm)}(u)]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,nm)}(u)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{nm}\widetilde{\Delta}_A(u)\right)\right) \\ &= m\zeta_A^{(nm)}([u]_1). \end{aligned}$$

We show the right triangle commutes. For $i \in \{0,1\}$, $u \in U_\infty(\mathbb{I}_n(A))$ and the inclusion $\iota : \mathbb{I}_n(A) \to \mathbb{I}_{nm}(A)$, $\mathrm{ev}_A^{(i,nm)}(\iota(u)) = \mathrm{ev}_A^{(i,n)}(u)$. Further as we work with non-normalised traces

$\widetilde{\Delta}_A(u \otimes 1_{M_m}) = m\widetilde{\Delta}(u)$. Combining the above fact with $\kappa_A^{(nm,n)}$ being the induced map of the inclusion,

$$\begin{aligned}\left(\zeta_A^{(nm)} \circ \kappa_A^{(nm,n)}\right)([u]_1) &= [\mathrm{ev}_A^{(0,nm)}(\iota(u))]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,nm)}(\iota(u))]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{nm}\widetilde{\Delta}_A(u \otimes 1_{M_m})\right) \\ &= [\mathrm{ev}_A^{(0,n)}(u)]_{\mathrm{alg}} - [\mathrm{ev}_A^{(1,nm)}(u)]_{\mathrm{alg}} + \mathrm{Th}_A\left(\frac{1}{n}\widetilde{\Delta}_A(u)\right) \\ &= \zeta_A^{(n)}([u]_1).\end{aligned}$$

□

We have now defined all the ingredients of the total invariant, and are ready to define and work with it.

6.2 The total invariant

After the hard work of the previous sections we are finally ready to define the total invariant for a unital C^*-algebra.

Definition 6.3 ([CGS$^+$23](3.5))**.** Let A be a unital C^*-algebra. The *total invariant* of A, denoted $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A)$ is

$$\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) := (\underline{K}(A), [1_A]_0, \mathrm{Aff}T(A), \overline{K}_1^{\mathrm{alg}}(A), \rho_A, \mathrm{Th}_A, \not{a}_A, (\zeta_A^{(n)})_{n\geq 2}),$$

where $\mathrm{Aff}T(A) = 0$ if $T(A) = \emptyset$ by convention.

A $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ morphism is a triple $(\underline{\alpha}, \gamma, \beta) : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$ where

- $\underline{\alpha} \in \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$ such that $\alpha_0([1_A]_0) = [1_B]_0$,
- $\gamma : \mathrm{Aff}T(A) \to \mathrm{Aff}T(B)$ is a positive linear map,
- $\beta : \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B)$ is a group homomorphism,

Where the morphisms satisfy

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \xrightarrow{\not{a}_A} & K_1(A) \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_1} \\ K_0(B) & \xrightarrow[\rho_B]{} & \mathrm{Aff}T(B) & \xrightarrow[\mathrm{Th}_B]{} & \overline{K}_1^{\mathrm{alg}}(B) & \xrightarrow[\not{a}_B]{} & K_1(B) \end{array}$$

and

$$\begin{array}{ccc} K_0(A;\mathbb{Z}_n) & \xrightarrow{\zeta_A^{(n)}} & \overline{K}_1^{\mathrm{alg}}(A) \\ \downarrow{\scriptstyle \alpha_0^{(n)}} & & \downarrow{\scriptstyle \beta} \\ K_0(B;\mathbb{Z}_n) & \xrightarrow[\zeta_B^{(n)}]{} & \overline{K}_1^{\mathrm{alg}}(B) \end{array}$$

commute for all $n \geq 2$.

Unital *-homomorphisms actually gives rise to $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ morphisms, in the obvious way, and thus $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ is a functor.

Proposition 6.4. *Let A, B be unital C^*-algebras and $\varphi : A \to B$ be a unital *-homomorphism. Then*

$$\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\varphi) := (\underline{K}(\varphi), \overline{K}_1^{\mathrm{alg}}(\varphi), \mathrm{Aff}(T(\varphi))) : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B).$$

is a $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}$ morphism.

Proof. We show

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \xrightarrow{\not{a}_A} & K_1(A) \\ {\scriptstyle K_0(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{Aff}(T(\varphi))} & & \downarrow{\scriptstyle \overline{K}_1^{\mathrm{alg}}(\varphi)} & & \downarrow{\scriptstyle K_1(\varphi)} \\ K_0(B) & \xrightarrow[\rho_B]{} & \mathrm{Aff}T(B) & \xrightarrow[\mathrm{Th}_B]{} & \overline{K}_1^{\mathrm{alg}}(B) & \xrightarrow[\not{a}_B]{} & K_1(B) \end{array}$$

commutes. First the left square. Let $p, q \in \mathcal{P}_n(A)$,

$$\begin{aligned} \mathrm{Aff}(T(\varphi))(\rho_A([p]_0 - [q]_0))(\tau) &= (\widehat{p}_n - \widehat{q}_n)(\tau \circ \varphi) \\ &= (\tau \circ \varphi)_n(p - q) \\ &= \tau_n(\varphi(p) - \varphi(q)) \\ &= \rho_B([\varphi(p)]_0 - [\varphi(q)]_0)(\tau) \\ &= \rho_B(K_0(\varphi)([p]_0 - [q]_0))(\tau). \end{aligned}$$

In the proof of Proposition 6.1 we see the central square commutes. To see the right square commute let $u \in U_\infty(A)$

$$K_1(\varphi)(\not{a}_A([u]_{\mathrm{alg}})) = K_1(\varphi)([u]_1) = [\varphi(u)]_1 = \not{a}_B([\varphi(u)]_{\mathrm{alg}}) = \not{a}_B(\overline{K}_1^{\mathrm{alg}}(\varphi)([u]_{\mathrm{alg}})).$$

The diagram

$$\begin{array}{ccc} K_0(A; \mathbb{Z}_n) & \xrightarrow{\zeta_A^{(n)}} & \overline{K}_1^{\mathrm{alg}}(A) \\ {\scriptstyle K_0^{(n)}(\varphi)}\downarrow & & \downarrow{\scriptstyle \overline{K}_1^{\mathrm{alg}}(\varphi)} \\ K_0(B; \mathbb{Z}_n) & \xrightarrow[\zeta_B^{(n)}]{} & \overline{K}_1^{\mathrm{alg}}(B) \end{array}$$

commutes by naturality of $\zeta^{(n)}$. □

We now move away from C^*-algebra land, and into the land of abstract nonsense, (category theory), where we will study the total invariant.

7 Splitting the total invariant

This section will be devoted to introducing the categories $\mathcal{KT}_u$-$\mathfrak{Sys}$ and $\underline{\mathcal{K}}\mathcal{T}_u$-$\mathfrak{Sys}$ representing an abstractification of the functors introduced in the previous sections. We will also show a splitting-esque result for an object of $\underline{\mathcal{K}}\mathcal{T}_u$-$\mathfrak{Sys}$.

7.1 Λ-Systems

In this subsection we will introduce the notion of a Λ-system and note that total K-theory is an example of this.

Definition 7.1. A Λ-*system*, $\mathbb{G}$ is a family of abelian groups and structure maps, more precisely

$$\mathbb{G} = ((G_j, G_{j,n}, \mu_n^j, \nu_n^j)_{j=0,1,n\geq 2}(\kappa_{n,m}^j, \kappa_{m,n}^j)_{j=0,1,n|m}),$$

where G_j are abelian groups, $G_{j,n}$ are $\mathbb{Z}_n$ modules, and $\mu_n^j, \nu_n^j, \kappa_{n,m}^j, \kappa_{m,n}^j$ are group homomorphisms such that

$$\begin{array}{ccccc} G_0 & \xrightarrow{\times n} & G_0 & \xrightarrow{\mu_n^0} & G_{0,n} \\ {\scriptstyle \nu_n^1}\uparrow & & & & \downarrow{\scriptstyle \nu_n^0} \\ G_{1,n} & \xleftarrow[\mu_n^1]{} & G_1 & \xleftarrow[\times n]{} & G_1 \end{array}$$

is a chain complex, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_j/nG_j & \xrightarrow{\overline{\mu}_n^j} & G_{j,n} & \xrightarrow{\overline{\nu}_n^j} & G_{1-j}[n] & \longrightarrow & 0 \\ & & {\scriptstyle \pi_{n,m}^j}\uparrow\downarrow{\scriptstyle \times\frac{m}{n}} & & {\scriptstyle \kappa_{n,m}^j}\uparrow\downarrow{\scriptstyle \kappa_{m,n}^j} & & {\scriptstyle \times\frac{m}{n}}\uparrow\downarrow{\scriptstyle \iota_{m,n}^{1-j}} & & \\ 0 & \longrightarrow & G_j/mG_j & \xrightarrow[\overline{\mu}_m^j]{} & G_{j,m} & \xrightarrow[\overline{\nu}_m^j]{} & G_{1-j}[m] & \longrightarrow & 0 \end{array}$$

commutes for $j = 0, 1$, $n|m$. Where $G[n] := \{g \in G \mid ng = 0\}$, $\overline{\mu}_n^j, \overline{\nu}_n^j$ are the induced homomorphisms, i.e $\mu_n^j = \overline{\mu}_n^j \circ \pi_n^j$ and $\nu_n^j = \iota_n^{1-j} \circ \overline{\nu}_n^j$, where $\pi_n^j : G_j \to G_j/nG_j$ is the projection and $\iota_n^{1-j} : G_{1-j}[n] \to G_{1-j}$ is the inclusion. Lastly we demand the rows are chains, i.e for all $\overline{\nu}_n^j \circ \overline{\mu}_n^j = 0$, $n, m > 2$. We say a Λ-system is exact if the chain complex above is exact, note that this implies the rows in the commuting diagram becomes short exact sequences. Sometimes we will abuse notation, and only write $\mathbb{G}$ and not the entire tuple, by which we will mean a Λ-system as described above if not otherwise stated. A Λ-*morphism*, $\alpha : \mathbb{G} \to \mathbb{H} = ((H_j, H_{j,n}, \sigma_n^j, \varphi_n^j)_{j=0,1,n\geq 2}(\psi_{n,m}^j, \psi_{m,n}^j)_{j=0,1,n|m})$, is a family of group homomorphisms

$$\alpha = (\alpha^j, \alpha_n^j)_{j=0,1,n\geq 2}$$

where $\alpha^j : G_j \to H_j$ and $\alpha_n^j : G_{j,n} \to H_{j,n}$ satisfies that the diagrams

$$\begin{array}{ccc} G_j & \xrightarrow{\alpha^j} & H_j \\ {\scriptstyle \mu_n^j}\downarrow & & \downarrow{\scriptstyle \sigma_n^j} \\ G_{j,n} & \xrightarrow[\alpha_n^j]{} & H_{j,n} \end{array} \qquad \begin{array}{ccc} G_{j,n} & \xrightarrow{\alpha_n^j} & H_{j,n} \\ {\scriptstyle \nu_n^j}\downarrow & & \downarrow{\scriptstyle \varphi_n^j} \\ G_{1-j} & \xrightarrow[\alpha^{1-j}]{} & H_{1-j} \end{array}$$

$$\begin{array}{ccccc} G_{j,n} & \xrightarrow{\kappa_{m,n}^j} & G_{j,m} & \xrightarrow{\kappa_{n,m}^j} & G_{j,n} \\ {\scriptstyle \alpha_n^j}\downarrow & & {\scriptstyle \alpha_m^j}\downarrow & & \downarrow{\scriptstyle \alpha_n^j} \\ H_{j,n} & \xrightarrow[\psi_{m,n}^j]{} & H_{j,m} & \xrightarrow[\psi_{n,m}^j]{} & H_{j,n} \end{array}$$

commutes. A Λ-isomorphism is a Λ-morphism where α^j and α_n^j are group isomorphisms for $j \in \{0, 1\}$ and $n \geq 2$. Λ-systems form a category which we will denote Λ-$\mathfrak{Sys}$.

Having seen this definition lets see some examples of exact Λ systems.

Example 7.2. Let A be a C^*-algebra, then the total K-theory $\underline{K}(A)$ is an exact Λ-system. This can be seen from the preliminaries section.

The next example will later be shown to be enough to describe all exact Λ-systems.

Example 7.3. Let G_0, G_1 be any pair of abelian groups. They induce the exact Λ-system $\Lambda(G_0, G_1)$ by defining for $n \geq 2$

$$G_{j,n} = (G_j/nG_j) \oplus G_{1-j}[n],$$

and the group homomorphisms

$$\begin{aligned} \mu_n^j &: G_j \to G_{j,n}, \quad x \mapsto ([x], 0), \\ \nu_n^j &: G_{j,n} \to G_{1-j} \quad ([x], y) \mapsto y, \end{aligned}$$

and whenever $n|m$

$$\begin{aligned} \kappa_{m,n}^j &: G_{j,n} \to G_{j,m}, \quad ([x], y) \mapsto \left(\left[\frac{m}{n}x\right], y\right), \\ \kappa_{n,m}^j &: G_{j,m} \to G_{j,n}, \quad ([x], y) \mapsto \left([x], \frac{m}{n}y\right). \end{aligned}$$

We call this the *trivial* Λ-system. Moreover given any morphism $(\alpha_0, \alpha_1) : G_0 \times G_1 \to H_0 \times H_1$, we can induce a Λ-morphism $\Lambda(\alpha_0, \alpha_1) = (\alpha^j, \alpha_n^j)_{j=0,1,n\geq 2} : \Lambda(G_0, G_1) \to \Lambda(H_0, H_1)$, where $\alpha^j = \alpha_j$ and α_n^j is given by

$$\alpha_n^j([x], y) = (\overline{\alpha}_n^j([x]), \widetilde{\alpha}_n^{1-j}(y)),$$

where $\overline{\alpha}_n^j$ and $\widetilde{\alpha}_n^{1-j}$ are the induced morphisms on G_j/nG_j and $G_{1-j}[n]$ respectively. Hence Λ is a functor $\Lambda : \mathfrak{Ab} \times \mathfrak{Ab} \to \Lambda\text{-}\mathfrak{Sys}$.

Proof. We show the 6-term sequence

$$\begin{array}{ccccc} G_0 & \xrightarrow{\times n} & G_0 & \xrightarrow{\mu_n^0} & G_0/nG_0 \oplus G_1[n] \\ \uparrow{\scriptstyle \nu_n^1} & & & & \downarrow{\scriptstyle \nu_n^0} \\ G_1/nG_1 \oplus G_0[n] & \xleftarrow[\mu_n^1]{} & G_1 & \xleftarrow[\times n]{} & G_1, \end{array}$$

is exact. We show exactness at $G_{j,n}$, note

$$\ker(\nu_n^j) = \{([x], 0) \in G_{j,n} \mid [x] \in G_j/nG_j\} = \mu_n^j(G_j).$$

We show exactness at the top left and bottom right corners. Observe

$$\ker(\times n) = G_j[n] = \nu_n^j(G_j/nG_j \oplus G_{1-j}[n]).$$

Lastly we show exactness at the middle groups. Notice

$$\ker(\mu_n^j) = nG_j.$$

We show the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_j/nG_j & \xrightarrow{\overline{\mu}_n^j} & G_j/nG_j \oplus G_j[n] & \xrightarrow{\overline{\nu}_n^j} & G_j[n] & \longrightarrow & 0 \\ & & {\scriptstyle \times 1}\uparrow\downarrow{\scriptstyle \times \frac{m}{n}} & & {\scriptstyle \kappa_{n,m}^j}\uparrow\downarrow{\scriptstyle \kappa_{m,n}^j} & & {\scriptstyle \times \frac{m}{n}}\uparrow\downarrow{\scriptstyle \times 1} & & \\ 0 & \longrightarrow & G_j/mG_j & \xrightarrow[\overline{\mu}_m^j]{} & G_j/mG_j \oplus G_j[m] & \xrightarrow[\overline{\nu}_m^j]{} & G_j[m] & \longrightarrow & 0, \end{array}$$

commutes. Note for all $n \geq 2$ the induced maps are given by $\overline{\mu}_n^j([x]) = ([x], 0)$ and $\overline{\nu}_n^j([x], y) = y$. We show the left square commute:

$$\left(\overline{\mu}_m^j \circ \times \frac{m}{n}\right)([x]) = \left(\left[\frac{m}{n}x\right], 0\right) = \kappa_{m,n}^j \circ \overline{\mu}_n^j([x]),$$
$$\kappa_{n,m}^j \circ \overline{\mu}_m^j([x]) = ([x], 0) = \mu_n^j([x]).$$

The right square

$$\overline{\nu}_m^j \circ \kappa_{m,n}^j([x], y) = y = \times 1 \circ \overline{\nu}_n^j$$
$$\overline{\nu}_n^j \circ \kappa_{n,m}^j([x], y) = \frac{m}{n} y = \times \frac{m}{n} \circ \overline{\nu}_m^j([x], y).$$

We show $\Lambda(\alpha_0, \alpha_1)$ is a Λ-morphism. First the diagrams

$$\begin{array}{ccc} G_j & \xrightarrow{\alpha^j} & H_j \\ {\scriptstyle \mu_n^j}\downarrow & & \downarrow{\scriptstyle \sigma_n^j} \\ G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow[\alpha_n^j]{} & H_j/nH_j \oplus H_{1-j}[n] \end{array} \qquad \begin{array}{ccc} G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow{\alpha_n^j} & H_j/nH_j \oplus H_{1-j}[n] \\ {\scriptstyle \nu_n^j}\downarrow & & \downarrow{\scriptstyle \varphi_n^j} \\ G_{1-j} & \xrightarrow[\alpha_n^{1-j}]{} & H_{1-j} \end{array}$$

commutes by definition of α_n^j. We show

$$\begin{array}{ccc} G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow{\alpha_n^j} & H_j/nH_j \oplus H_{1-j}[n] \\ {\scriptstyle \kappa_{m,n}^{G_j}}\downarrow & & \downarrow{\scriptstyle \kappa_{m,n}^{H_j}} \\ G_j/mG_j \oplus G_{1-j}[m] & \xrightarrow[\alpha_m^j]{} & H_j/mH_j \oplus H_{1-j}[m], \end{array}$$

commutes. Observe

$$\begin{aligned} \kappa_{m,n}^{H_j} \circ \alpha_n^j([x]_{G_j/nG_j}, y) &= \left(\times \frac{m}{n} \overline{\alpha}_n^j([x]_{G_j/nG_j}), \widetilde{\alpha}_n^{1-j}(y)\right) \\ &= \left(\times \frac{m}{n} [\alpha^j(x)]_{H_j/nH_j}, \widetilde{\alpha}_n^{1-j}(y)\right) \\ &= \left(\left[\alpha^j\left(\frac{m}{n}x\right)\right]_{H_j/mH_j}, \widetilde{\alpha}_n^{1-j}(y)\right) \\ &= \left(\overline{\alpha}_m^j\left(\left[\frac{m}{n}x\right]_{G_j/mG_j}\right), \widetilde{\alpha}_n^{1-j}(y)\right). \end{aligned}$$

Further $\widetilde{\alpha}_n^{1-j}(y) = \widetilde{\alpha}_m^{1-j}(y)$ indeed for the diagram

$$\begin{array}{ccccc} & G_j & \xrightarrow{\alpha^j} & H_j & \\ {\scriptstyle \iota_m^{G_j}}\uparrow & \uparrow{\scriptstyle \iota_n^{G_j}} & & \uparrow{\scriptstyle \iota_n^{H_j}} & \uparrow{\scriptstyle \iota_m^{H_j}} \\ & G_j[n] & \xrightarrow{\widetilde{\alpha}_n^j} & H_j[n] & \\ & \downarrow{\scriptstyle \iota_{m,n}^{G_j}} & & \downarrow{\scriptstyle \iota_{m,n}^{H_j}} & \\ & G_j[m] & \xrightarrow[\alpha_m^j]{} & H_j[m] & \end{array}$$

all but the bottom square commutes. Hence

$$\iota_m^{H_j} \circ \iota_{m,n}^{H_j} \circ \widetilde{\alpha}_n^j = \iota^{H_j} \circ \widetilde{\alpha}_n^j = \alpha_j \circ \iota_n = \alpha^j \circ \iota_m^{G_j} \circ \iota_{m,n}^{G_j} = \iota_m^{H_j} \circ \widetilde{\alpha}_m^j \circ \iota_{m,n}^{G_j}.$$

By injectivity of $\iota_m^{H_j}$, the bottom square commutes. Continuing our computation

$$\left(\overline{\alpha}_m^j\left(\left[\frac{m}{n}x\right]_{G_j/mG_j}\right), \widetilde{\alpha}_n^{1-j}(y)\right) = \left(\overline{\alpha}_m^j\left(\left[\frac{m}{n}x\right]_{G_j/mG_j}\right), \widetilde{\alpha}_m^{1-j}(y)\right) = \alpha_m^j \circ \kappa_{m,n}^{G_j}([x], y).$$

We show the diagram

$$\begin{array}{ccc} G_j/mG_j \oplus G_{1-j}[m] & \xrightarrow{\alpha_m^j} & H_j/mH_j \oplus H_{1-j}[m] \\ \kappa_{n,m}^{G_j}\downarrow & & \downarrow\kappa_{n,m}^{H_j} \\ G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow[\alpha_n^j]{} & H_j/nH_j \oplus H_{1-j}[n], \end{array}$$

commutes.

$$\kappa_{n,m}^{H_j} \circ \alpha_m^j([x]_{G_j/mG_j}, y) = \left(\overline{\alpha}_m^j([x]_{G_j/mG_j}), \widetilde{\alpha}_m^{1-j}\left(\frac{m}{n}y\right)\right),$$
$$\alpha_n^j \circ \kappa_{n,m}^{G_j}([x]_{G_j/mG_j}, y) = \left(\overline{\alpha}_n^j([x]_{G_j/nG_j}), \widetilde{\alpha}_n^{1-j}\left(\frac{m}{n}y\right)\right).$$

As $\times 1 : G_j/mG_j \to G_j/nG_j$ is well defined $\overline{\alpha}_m^j([x]_{G_j/mG_j}) = \overline{\alpha}_n^j([x]_{G_j/nG_j})$ and our previous calculations shows that $\widetilde{\alpha}_m^{1-j}\left(\frac{m}{n}y\right) = \widetilde{\alpha}_n^{1-j}\left(\frac{m}{n}y\right)$, hence the diagram commutes. □

The next result is due to Bödigheimer in [Bö79], [Bö80] where he proved it whenever n, m are powers of the same prime. Using the Chinese remainder theorem this can be extended to general n, m, for our purpose it suffices to show this when n divides m or m divides n, a the proof of this extension can be found in appendix A.3 of [CGS$^+$23]. Further the result has been reformulated in terms of Λ-systems and we will only state the result, as after this statement we will prove a reformulation.

Proposition 7.4 ([Bö79], [Bö80], [CGS$^+$23]). *Let $\mathbb{G}$ be an exact Λ-system. Then for $j \in \{0,1\}$ and $n \geq 2$ there are (unnaturally chosen) homomorphisms*

$$s_n^j : G_{1-j}[n] \to G_{j,n},$$

such that

1. $s_n^j \circ \nu_n^j = \mathrm{id}_{G_{1-j}[n]}$.
2. $s_m^j \circ \iota_{m,n}^{1-j} = \kappa_{m,n} \circ s_n^j$ *for all $n, m \geq 2$ and $n \mid m$ or $m \mid n$.*
3. $\kappa_{n,m}^j \circ s_m = s_n \circ \times\frac{m}{n}$ *for all $n, m \geq 2$ and $n \mid m$ or $m \mid n$.*

Before we can state the reformulation of Bödigheimer, we first specify what the forgetful functor is.

Remark 7.5. The forgetful functor $F_\Lambda : \Lambda\text{-}\mathfrak{sys} \to \mathfrak{Ab} \times \mathfrak{Ab}$ is given on objects by $F_\Lambda(\mathbb{G}) = (G_0, G_1)$ and on morphisms by $F_\Lambda(\alpha) = (\alpha^0, \alpha^1)$.

Now for the reformulation of Bödigheimer.

Theorem 7.6 (Bödigheimer)**.** *Every exact Λ-system $\mathbb{G}$ is (unnaturally) isomorphic to the trivial Λ-system of $F_\Lambda(\mathbb{G})$, $\Lambda(F_\Lambda(\mathbb{G}))$.*

Proof. Fix the homomorphisms s_n^j from Proposition 7.4, and define for $j \in \{0,1\}$ and $n \geq 2$ the homomorphisms $\varphi_{j,n} : (G_j/nG_j) \oplus G_{1-j}[n] \to G_{j,n}$ by

$$\varphi_{j,n}([x], y) = \mu_n^j(x) + s_n^j(y).$$

We show $(\mathrm{id}_{G_j}, \varphi_{j,n})_{j\in\{0,1\},n\geq 2}$ is a Λ-morphism. For the rest of the proof denote by $\pi_{j,n}^1 : G_j \to G_j/nG_j \oplus G_{1-j}[n]$ the homomorphism $\pi_{j,n}^1(x) = ([x], 0)$ and $\iota_{j,n}^2 : G_j/nG_j \oplus G_{1-j}[n] \to G_{1-j}[n]$ the homomorphism $\iota_{j,n}^2([x], y) = y$. The diagram

$$\begin{array}{ccc} G_j & = & G_j \\ \downarrow{\scriptstyle \pi_{j,n}^1} & & \downarrow{\scriptstyle \mu_n^j} \\ G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow[\varphi_{j,n}]{} & G_{j,n}, \end{array}$$

commutes since

$$\varphi_{j,n} \circ \pi_{j,n}^1(x) = \varphi_{j,n}([x], 0) = \mu_n^j(x).$$

Next diagram

$$\begin{array}{ccc} G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow{\varphi_{j,n}} & G_{j,n} \\ \downarrow{\scriptstyle \iota_{j,n}^2} & & \downarrow{\scriptstyle \nu_n^j} \\ G_{j-1} & = & G_{j-1}, \end{array}$$

commutes as

$$\nu_n^j \circ \varphi_{j,n}([x], y) = \underbrace{\nu_n^j(\mu(x))}_{=0} + \nu_n^j(s_n^j(y)) = y = \iota_{j,n}^2([x], y).$$

We show the diagram

$$\begin{array}{ccccc} G_j/nG_j \oplus G_{1-j}[n] & \xrightarrow{K_{m,n}^j} & G_j/mG_j \oplus G_{1-j}[m] & \xrightarrow{K_{n,m}^j} & G_j/nG_j \oplus G_{1-j}[n] \\ \downarrow{\scriptstyle \varphi_{j,n}} & & \downarrow{\scriptstyle \varphi_{j,m}} & & \downarrow{\scriptstyle \varphi_{j,n}} \\ G_{j,n} & \xrightarrow[\kappa_{m,n}^j]{} & G_{j,m} & \xrightarrow[\kappa_{n,m}^j]{} & G_{j,n}, \end{array}$$

commutes. Start with the left square, by Proposition 7.4

$$\begin{aligned} \kappa_{m,n}^j \circ \varphi_{j,n}([x], y) &= \kappa_{m,n}^j \circ \mu_n^j(x) + \kappa_{m,n}^j \circ s_n(y) \\ &= \kappa_{m,n}^j \circ \overline{\mu}_n^j([x]) + s_m \circ \iota_{m,n}^{1-j}(y) \\ &= \overline{\mu}_m^j \circ \times \frac{m}{n}([x]) + s_m(y) \\ &= \mu_m^j\left(\frac{m}{n}x\right) + s_m(y) \\ &= \varphi_{j,m}(K_{m,n}^j([x], y)). \end{aligned}$$

Next the right square, again using Proposition 7.4,

$$\begin{aligned}\kappa^j_{n,m}\circ\varphi_{j,m}([x],y) &= \kappa^j_{n,m}\circ\mu^j_m(x)+\kappa^j_{n,m}\circ s^j_m(y)\\ &= \kappa^j_{n,m}\circ\overline{\mu}^j_m([x])+s_n\circ\times\frac{m}{n}(y)\\ &= \overline{\mu}^j_n([x])+s_n\circ\times\frac{m}{n}(y)\\ &= \varphi_{j,n}(K^j_{n,m}([x],y)).\end{aligned}$$

We show $(\varphi_{j,n})_{j\in\{0,1\},n\geq 2}$ are group isomorphisms, which implies $(\mathrm{id}_{G_j},\varphi_{j,n})_{j\in\{0,1\},n\geq 2}$ is a Λ-isomorphism. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_j/nG_j & \xrightarrow{\overline{\pi}^1_{j,n}} & G_j/nG_j\oplus G_{1-j}[n] & \xrightarrow{\overline{\iota}^2_{j,n}} & G_{1-j}[n] & \longrightarrow & 0\\ & & \| & & \downarrow{\scriptstyle\varphi_{j,n}} & & \| & & \\ 0 & \longrightarrow & G_j/nG_j & \xrightarrow[\overline{\mu}^j_n]{} & G_{j,n} & \xrightarrow[\overline{\nu}^j_n]{} & G_{1-j}[n] & \longrightarrow & 0\end{array}$$

As $\overline{\pi}^1_{j,n}([x])=([x],0)$ and $\overline{\iota}^2_{j,n}([x],y)=y$ our calculation above shows $\varphi_{j,n}$ makes the diagram commute, hence $\varphi_{j,n}$ is a group isomorphism by the the 5-lemma. ☐

Remark:
For the rest of the thesis whenever we say something follows from Bödigheimer it will be with Theorem 7.6 in mind, unless stated otherwise.

We see we can apply Bödigheimer to total K-theory, which would tell us, the total K-theory of a C^*-algebra is determined by it's K-theory.

Corollary 7.7. *Let A be a C^*-algebra, then there is an unnatural isomorphism*

$$\underline{K}(A)\cong\Lambda(K_0(A),K_1(A)).$$

Proof. By Example 7.2 $\underline{K}(A)$ is an exact Λ-system and $F_\Lambda(\underline{K}(A))=(K_0(A),K_1(A))$. Applying Bödigheimer (7.6) we get the desired isomorphism. ☐

We now look at bit closer at the forgetful functor F_Λ.

Definition 7.8. Let $\mathbb{G}$ and $\mathbb{H}$ be Λ-systems, the forgetful functor F_Λ induces a homomorphism

$$\widetilde{F_\Lambda}:\mathrm{Hom}_\Lambda(\mathbb{G},\mathbb{H})\to\mathrm{Hom}(F_\Lambda(\mathbb{G}),F_\Lambda(\mathbb{H})).$$

Let $\mathcal{N}(\mathbb{G},\mathbb{H})$ denote the kernel of this homomorphism.

Bödigheimer also tells us more about this situation.

Corollary 7.9. *Let $\mathbb{G}$ and $\mathbb{H}$ be exact Λ-systems, then the sequence*

$$0\longrightarrow\mathcal{N}_\Lambda(\mathbb{G},\mathbb{H})\overset{\iota_\Lambda}{\rightarrowtail}\mathrm{Hom}_\Lambda(\mathbb{G},\mathbb{H})\xrightarrow{\widetilde{F_\Lambda}}\mathrm{Hom}(F_\Lambda(\mathbb{G}),F_\Lambda(\mathbb{H}))\longrightarrow 0,$$

is split exact (unnaturally).

Proof. As ι_Λ is the inclusion we get exactness at $\mathcal{N}_\Lambda(\mathbb{G},\mathbb{H})$ and $\iota_\Lambda(\mathcal{N}_\Lambda(\mathbb{G},\mathbb{H})) = \mathcal{N}_\Lambda(\mathbb{G},\mathbb{H}) =: \ker(\widetilde{F}_\Lambda)$ showing exactness at $\mathrm{Hom}_\Lambda(\mathbb{G},\mathbb{H})$. Surjectivity of $\widetilde{F}_\Lambda$ follows from Bödigheimer, indeed let $\Phi_\mathbb{G} : \Lambda(F_\Lambda(\mathbb{G})) \to \mathbb{G}$ and $\Phi_\mathbb{H} : \Lambda(F_\Lambda(\mathbb{H})) \to \mathbb{H}$ be the isomorphisms from Bödigheimer, 7.6. Let $(\alpha_0,\alpha_1) : F_\Lambda(\mathbb{G}) \to F_\Lambda(\mathbb{H})$ be a homomorphism then $\Phi_\mathbb{H} \circ \Lambda(\alpha_0,\alpha_1) \circ \Phi_\mathbb{G}^{-1}$ defines a Λ-morphism from $\mathbb{G}$ to $\mathbb{H}$, moreover

$$\widetilde{F}_\Lambda(\Phi_\mathbb{H} \circ \Lambda(\alpha_0,\alpha_1) \circ \Phi_\mathbb{G}^{-1}) = \widetilde{F}_\Lambda(\Phi_\mathbb{H}) \circ \widetilde{F}_\Lambda(\Lambda(\alpha_0,\alpha_1)) \circ \widetilde{F}_\Lambda(\Phi_\mathbb{G}^{-1}) = (\alpha_0,\alpha_1).$$

Lastly let $\Psi : \mathrm{Hom}(F_\Lambda(\mathbb{G}), F_\Lambda(\mathbb{H})) \to \mathrm{Hom}_\Lambda(\mathbb{G},\mathbb{H})$ be given by

$$\Psi(\alpha_0,\alpha_1) = \Phi_\mathbb{H} \circ \Lambda(\alpha_0,\alpha_1) \circ \Phi_\mathbb{G}^{-1}.$$

By the above calculation $\widetilde{F}_\Lambda \circ \Psi = \mathrm{id}_{\mathrm{Hom}(F_\Lambda(\mathbb{G}),F_\Lambda(\mathbb{H}))}$. □

The last result of this subsection will be to describe $\mathcal{N}(\mathbb{G},\mathbb{H})$ precisely.

Proposition 7.10. *Let $\mathbb{G}$ and $\mathbb{H}$ be exact Λ-systems, then there is a natural bijection between $\mathcal{N}(\mathbb{G},\mathbb{H})$ and the set of families of maps*

$$(\varphi_{j,n} : G_{1-j}[n] \to H_j/nH_j)_{j=0,1,n\geq 2}$$

such that all the diagrams of the following form commutes for $n|m$,

$$\begin{array}{ccccc} G_{1-j}[n] & \xrightarrow{\iota_{m,n}^{G_{1-j}}} & G_{1-j}[m] & \xrightarrow{\times \frac{n}{m}} & G_{1-j}[n] \\ \downarrow{\scriptstyle \varphi_{j,n}} & & \downarrow{\scriptstyle \varphi_{j,m}} & & \downarrow{\scriptstyle \varphi_{j,n}} \\ H_j/nH_j & \xrightarrow{\times \frac{m}{n}} & H_j/mH_j & \xrightarrow{\pi_{n,m}^{H_j}} & H_j/nH_j. \end{array}$$

We will denote this family $\mathcal{D}(\mathbb{G},\mathbb{H})$.

Proof. Define a map $\Phi_{\mathbb{G},\mathbb{H}} : \mathcal{D}(\mathbb{G},\mathbb{H}) \to \mathcal{N}(\mathbb{G},\mathbb{H})$ by

$$\Phi((\varphi_{j,n})_{j=0,1,n\geq 2}) = (0, \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j})_{j=0,1,n\geq 2}.$$

We show $\Phi_{\mathbb{G},\mathbb{H}}$ is well defined, that is we show $(0, \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j})_{j=0,1,n\geq 2}$ is a Λ-morphism. We show

$$\begin{array}{ccc} G_j & \xrightarrow{0} & H_j \\ \downarrow{\scriptstyle \mu_n^{G_j}} & & \downarrow{\scriptstyle \mu_n^{H_j}} \\ G_{j,n} & \xrightarrow{\overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j}} & H_{j,n} \end{array}$$

commutes. Observe

$$\overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} \circ \mu_n^{G_j} = \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \underbrace{\overline{\nu}_n^{G_j} \circ \overline{\mu}_n^{G_j}}_{=0} \circ \pi_n^j = 0.$$

Next

$$\begin{array}{ccc} G_{j,n} & \xrightarrow{\overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j}} & H_{j,n} \\ \downarrow{\scriptstyle \nu_n^{G_j}} & & \downarrow{\scriptstyle \nu_n^{H_j}} \\ G_{1-j} & \xrightarrow[0]{} & H_{1-j}, \end{array}$$

commutes, as
$$\nu_n^{H_j} \circ \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} = \iota_n^{1-j} \circ \underbrace{\overline{\nu}_n^{H_j} \circ \overline{\mu}_n^{H_j}}_{=0} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} = 0.$$

We show
$$\begin{array}{ccccc} G_{j,n} & \xrightarrow{\kappa_{m,n}^{G_j}} & G_{j,m} & \xrightarrow{\kappa_{n,m}^{G_j}} & G_{j,n} \\ \downarrow{\scriptstyle \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j}} & & \downarrow{\scriptstyle \overline{\mu}_m^{H_j} \circ \varphi_{j,m} \circ \overline{\nu}_m^{G_j}} & & \downarrow{\scriptstyle \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j}} \\ H_{j,n} & \xrightarrow[\kappa_{m,n}^{H_j}]{} & H_{j,m} & \xrightarrow[\kappa_{n,m}^{H_j}]{} & H_{j,n}, \end{array}$$

commutes. We start with the left square, as $\mathbb{G}$ and $\mathbb{H}$ are Λ-systems
$$\begin{aligned} \kappa_{m,n}^{H_j} \circ \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} &= \overline{\mu}_m^{H_j} \circ \times \frac{m}{n} \circ \varphi_{j,m} \circ \overline{\nu}_n^{G_j} \\ &= \overline{\mu}_m^{H_j} \circ \varphi_{j,m} \circ \iota_{m,n}^{G_j} \circ \overline{\nu}_n^{G_j} \\ &= \overline{\mu}_m^{H_j} \circ \varphi_{j,m} \circ \overline{\nu}_m^{G_j} \circ \kappa_{m,n}^{G_j}. \end{aligned}$$

Next the right square,
$$\begin{aligned} \kappa_{n,m}^{H_j} \circ \overline{\mu}_m^{H_j} \circ \varphi_{j,m} \circ \overline{\nu}_m^{G_j} &= \overline{\mu}_n^{H_j} \circ \pi_{n,m}^{H_j} \circ \varphi_{j,m} \circ \overline{\nu}_m^{G_{1-j}} \\ &= \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \times \frac{m}{n} \circ \overline{\nu}_m^{G_j} \\ &= \overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} \circ \kappa_{n,m}^{G_j}. \end{aligned}$$

We show that $\Phi_{\mathbb{G},\mathbb{H}}$ is injective. Let $(\varphi_{j,n})_{j=0,1,n\geq 2}, (\psi_{j,n})_{j=0,1,n\geq 2} \in \mathcal{D}(\mathbb{G},\mathbb{H})$, and assume
$$\Phi_{\mathbb{G},\mathbb{H}}((\varphi_{j,n})_{j=0,1,n\geq 2}) = \Phi_{\mathbb{G},\mathbb{H}}((\psi_{j,n})_{j=0,1,n\geq 2})$$

or equivalently for $j \in \{0,1\}$ and for all $n \geq 2$,
$$\overline{\mu}_n^{H_j} \circ \varphi_{j,n} \circ \overline{\nu}_n^{G_j} = \overline{\mu}_n^{H_j} \circ \psi_{j,n} \circ \overline{\nu}_n^{G_j}.$$

Since $\overline{\mu}_n^{H_j}$ is injective and $\overline{\nu}_n^{G_j}$ is surjective for $j \in \{0,1\}$ and for all $n \geq 2$
$$\varphi_{j,n} = \psi_{j,n}.$$

We show $\Phi_{\mathbb{G},\mathbb{H}}$ is surjective. Let $f = (0, \alpha_n^j)$ be an element of $\mathcal{N}(\mathbb{G},\mathbb{H})$, hence the diagram with exact rows
$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_j/nG_j & \xrightarrow{\overline{\mu}_n^{G_j}} & G_{j,n} & \xrightarrow{\overline{\nu}_n^{G_j}} & G_{1-j}[n] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_n^j} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & H_j/nH_j & \xrightarrow{\overline{\mu}_n^{H_j}} & H_{j,n} & \xrightarrow{\overline{\nu}_n^{H_j}} & H_{1-j}[n] & \longrightarrow & 0, \end{array}$$

commutes. We wish to construct a family of maps $\Delta_{j,n} : G_{1-j}[n] \to H_j/nH_j$ such that $\Delta_{j,n} \in \mathcal{D}(\mathbb{G},\mathbb{H})$ and $\Phi((\Delta_{j,n})_{j=0,1,n\geq 2}) = (0, \alpha_n^j)$. As $\overline{\nu}_n^{G_j}$ is surjective, for $x' \in G_{1-j}[n]$ pick some $x \in G_{j,n}$ such that $\overline{\nu}_n^{G_j}(x) = x'$, as the diagram commutes
$$\overline{\nu}_n^{H_j}(\alpha_n^j) = 0.$$

So $\mathrm{Im}(\alpha_n^j) \subseteq \ker\left(\overline{\nu}_n^{H_j}\right) = \mathrm{Im}\left(\overline{\mu}_n^{H_j}\right)$, hence there exists $[h]_n \in H_j/nH_j$ such that $\overline{\mu}_n^{H_j}([h]_n) = \alpha_n^j(x)$. Define $\Delta_{j,n} : G_{1-j}[n] \to H_j/nH_j$ by $\Delta(x') = [h]_n$ and by construction $\alpha_n^j = \overline{\mu}_n^{H_j} \circ \Delta_{j,n} \circ \overline{\nu}_n^{G_j}$, hence if $\Delta_{j,n} \in \mathcal{D}(\mathbb{G},\mathbb{H})$ then $\Phi_{\mathbb{G},\mathbb{H}}$ is surjective. We show the diagram

$$\begin{array}{ccccc}
G_{1-j}[n] & \xrightarrow{\iota_{m,n}^{G_{1-j}}} & G_{1-j}[m] & \xrightarrow{\times\frac{m}{n}} & G_{1-j}[n] \\
\Big\downarrow{\scriptstyle \Delta_{j,n}} & & \Big\downarrow{\scriptstyle \Delta_{j,m}} & & \Big\downarrow{\scriptstyle \Delta_{j,n}} \\
H_j/nH_j & \xrightarrow{\times\frac{m}{n}} & H_j/mH_j & \xrightarrow{\pi_{n,m}^{H_j}} & H_j/nH_j
\end{array}$$

commutes. First show the left square commutes. Since $(0,\alpha_n^j)_{j=0,1,n\geq 2}$ is a Λ-morphism

$$\begin{aligned}
\overline{\mu}_m^{H_j} \circ \times\frac{m}{n} \circ \Delta_{j,n} \circ \overline{\nu}_n^{G_j} &= \kappa_{m,n}^{H_j} \circ \overline{\mu}_n^j \circ \Delta_{j,n} \circ \overline{\nu}_n^{G_j} \\
&= \kappa_{m,n}^{H_j} \circ \alpha_n^j \\
&= \alpha_m^j \circ \kappa_{m,n}^{G_j} \\
&= \overline{\mu}_m^j \circ \Delta_{j,m} \circ \overline{\nu}_m^{G_j} \circ \kappa_{m,n}^{G_j} \\
&= \overline{\mu}_m^j \circ \Delta_{j,m} \circ \iota_{m,n}^{G_{1-j}} \circ \overline{\nu}_n^{G_j}.
\end{aligned}$$

Again using that $\overline{\mu}_m^j$ is injective and $\overline{\nu}_n^j$ is surjective $\times\frac{m}{n} \circ \Delta_{j,n} = \Delta_{j,m} \circ \iota_{m,n}^{G_j}$. We show the right square commutes, again as $(0,\alpha_n^j)_{j=0,1,n\geq 2}$ is a Λ-morphism

$$\begin{aligned}
\overline{\mu}_n^{H_j} \circ \Delta_{j,n} \circ \times\frac{m}{n} \circ \overline{\nu}_m^{G_j} &= \overline{\mu}_n^{H_j} \circ \Delta_{j,n} \circ \overline{\nu}_n^{G_j} \circ \kappa_{n,m}^{G_j} \\
&= \alpha_n^j \circ \kappa_{n,m}^{G_j} \\
&= \kappa_{n,m}^{H_j} \circ \alpha_m^j \\
&= \kappa_{n,m}^{H_j} \circ \overline{\mu}_m^{H_j} \circ \Delta_{j,m} \circ \overline{\nu}_m^{G_j} \\
&= \overline{\mu}_n^{H_j} \circ \pi_{n,m}^{H_j} \circ \Delta_{j,m} \circ \overline{\nu}_m^{G_j},
\end{aligned}$$

and as $\overline{\mu}_n^{H_j}$ is injective and $\overline{\nu}_m^{G_j}$ is surjective $\Delta_{j,n} \circ \times\frac{m}{n} = \pi_{n,m}^{H_j} \circ \Delta_{j,m}$. The last thing to show is naturality. Note given a Λ-morphism $(\beta^j, \beta_n^j)_{j=0,1,n\geq 2} = \Lambda_G : \mathbb{G}_1 \to \mathbb{G}_2$ we get induced morphisms

$$\overline{\Lambda}_G : \mathcal{D}(\mathbb{G}_2,\mathbb{H}) \to \mathcal{D}(\mathbb{G}_1,\mathbb{H}), \quad \widetilde{\Lambda}_G : \mathcal{N}(\mathbb{G}_2,\mathbb{H}) \to \mathcal{N}(\mathbb{G}_1,\mathbb{H}),$$

given by

$$\overline{\Lambda}_G((\varphi_{j,n})_{j=0,1,n\geq 2}) = (\varphi_{j,n} \circ \widetilde{\beta}^{1-j})_{j=0,1,n\geq 2}, \text{ and } \widetilde{\Lambda}_G((0,\alpha_n^j)_{j=0,1,n\geq 2}) = (0, \alpha_n^j \circ \beta_n^j)_{j=0,1,n\geq 2}.$$

Given a Λ-morphism $(\gamma^j, \gamma_n^j)_{j=0,1,n\geq 2} = \Lambda_H : \mathbb{H}_1 \to \mathbb{H}_2$ we get induced morphisms

$$\widetilde{\Lambda}_H : \mathcal{D}(\mathbb{G},\mathbb{H}_1) \to \mathcal{D}(\mathbb{G},\mathbb{H}_2) \quad \widetilde{\Lambda}_H : \mathcal{N}(\mathbb{G},\mathbb{H}_1) \to \mathcal{N}(\mathbb{G},\mathbb{H}_2)$$

given by

$$\overline{\Lambda}_H((\varphi_{j,n})_{j=0,1,n\geq 2}) = (\overline{\gamma}^j \circ \varphi_{j,n})_{j=0,1,n\geq 2}, \text{ and } \widetilde{\Lambda}_H((0,\alpha_n^j)_{j=0,1,n\geq 2}) = (0, \gamma_n^j \circ \alpha_n^j)_{j=0,1,n\geq 2}.$$

$\Phi_{-,-}$ is natural if

$$\begin{array}{ccc}
\mathcal{D}(\mathbb{G}_2,\mathbb{H}) & \xrightarrow{\overline{\Lambda}_G} & \mathcal{D}(\mathbb{G}_1,\mathbb{H}) \\
\Big\downarrow{\scriptstyle \Phi_{\mathbb{G}_2,\mathbb{H}}} & & \Big\downarrow{\scriptstyle \Phi_{\mathbb{G}_1,\mathbb{H}}} \\
\mathcal{N}(\mathbb{G}_2,\mathbb{H}) & \xrightarrow{\widetilde{\Lambda}_G} & \mathcal{N}(\mathbb{G}_1,\mathbb{H})
\end{array}
\quad , \quad
\begin{array}{ccc}
\mathcal{D}(\mathbb{G},\mathbb{H}_1) & \xrightarrow{\overline{\Lambda}_H} & \mathcal{D}(\mathbb{G},\mathbb{H}_2) \\
\Big\downarrow{\scriptstyle \Phi_{\mathbb{G},\mathbb{H}_1}} & & \Big\downarrow{\scriptstyle \Phi_{\mathbb{G},\mathbb{H}_2}} \\
\mathcal{N}(\mathbb{G},\mathbb{H}_1) & \xrightarrow{\widetilde{\Lambda}_H} & \mathcal{N}(\mathbb{G},\mathbb{H}_2)
\end{array}$$

commutes. We show the left square commutes.

$$\Phi_{\mathbb{G}_1,\mathbb{H}}(\overline{\Lambda}_G((\varphi_{j,n})_{j=0,1,n\geq 2}) = \Phi_{\mathbb{G}_1,\mathbb{H}}((\varphi_{j,n}\circ\widetilde{\beta}^{1-j})_{j=0,1,n\geq 2}) = (0,\overline{\mu}_n^{H_j}\circ\varphi_{j,n}\circ\widetilde{\beta}^{1-j}\circ\overline{\nu}_n^{G_j^1})_{j=0,1,n\geq 2}).$$

Then as (β^j, β_n^j) is a Λ-morphism

$$\nu_n^{G_j^1}\beta^{1-j} = \beta_n^j \nu_n^{G_j^2},$$

which also holds for the induced map on the n-torsion subgroup, continuing our calculation

$$\begin{aligned}
(0,\overline{\mu}_n^{H_j}\circ\varphi_{j,n}\circ\widetilde{\beta}^{1-j}\circ\overline{\nu}_n^{G_j^1})_{j=0,1,n\geq 2}) &= (0,\overline{\mu}_n^{H_j}\circ\varphi_{j,n}\circ\overline{\nu}_n^{G_j^2}\circ\beta_n^j)_{j=0,1,n\geq 2}) \\
&= \widetilde{\Lambda}_G((0,\overline{\mu}_n^j\circ\varphi_{j,n}\circ\overline{\nu}_n^{G_j^2})_{j=0,1,n\geq 2}) \\
&= \widetilde{\Lambda}_G(\Phi_{\mathbb{G}_2,\mathbb{H}}(\varphi_{j,n})_{j=0,1,n\geq 2}).
\end{aligned}$$

We show the right square commutes.

$$\Phi_{\mathbb{G},\mathbb{H}_2}(\overline{\Lambda}_H((\varphi_{j,n})_{j=0,1,n\geq 2})) = \Phi_{\mathbb{G},\mathbb{H}_2}((\overline{\gamma}^j\circ\varphi_{j,n})_{j=0,1,n\geq 2}) = (0,\overline{\mu}_n^{H_j^2}\circ\overline{\gamma}^j\circ\varphi_{j,n}\circ\nu_n^{G_j})_{j=0,1,n\geq 2}.$$

As before we use that $(\gamma^j, \gamma_n^j)_{j=0,1,n\geq 2}$ is a Λ-morphism to get

$$\mu_n^{H_j^2}\gamma^j = \gamma_n^j \mu_n^{H_j^1},$$

which also holds for the induced quotient maps, continuing

$$\begin{aligned}
(0,\overline{\mu}_n^{H_j^2}\circ\overline{\gamma}^j\circ\varphi_{j,n}\circ\nu_n^{G_j})_{j=0,1,n\geq 2} &= (0,\gamma_n^j\circ\overline{\mu}_n^{H_j^1}\circ\varphi_{j,n}\circ\nu_n^{G_j})_{j=0,1,n\geq 2} \\
&= \widetilde{\Lambda}_H(\overline{\mu}_n^{H_j^1}\circ\varphi_{j,n}\circ\nu_n^{G_j})_{j=0,1,n\geq 2}) \\
&= \widetilde{\Lambda}_H(\Phi_{\mathbb{G},\mathbb{H}_1}).
\end{aligned}$$

□

7.2 $\underline{\mathcal{K}}\mathcal{T}_u$-Systems

In this section we will be inspired by the total invariant as well as the results in the previous section, to attempt similar constructions for the total invariant. We start by abstractly defining what would be the total invariant.

Definition 7.11. A $\underline{\mathcal{K}}\mathcal{T}_u$-*system* is an octuple

$$(\mathbb{G}, g, X, D, \xi, \upsilon, \psi, (\zeta_n)_{n\geq 2}),$$

where $\mathbb{G} = ((G_j, G_{j,n}, \mu_n^j, \nu_n^j)_{j=0,1,n\geq 2}(\kappa_{n,m}^j, \kappa_{m,n}^j)_{j=0,1,n|m})$ is a Λ-system, $g \in G_0$, X is an order unit space, D is an abelian group, $\xi : G_0 \to X$ is a group homomorphism such that $\xi(g) = e$ the order unit in X, and $\upsilon : X \to D$, $\psi : D \to G_1$, $\zeta_n : G_{0,n} \to D$ are all group homomorphisms, making the diagram commute

$$\begin{array}{ccccc}
G_0 & \xrightarrow{\mu_n^0} & G_{0,n} & \xrightarrow{\nu_n^0} & G_1 \\
{\scriptstyle \frac{1}{n}\xi}\downarrow & & {\scriptstyle \zeta_n}\downarrow & & \Vert \\
X & \xrightarrow[\upsilon]{} & D & \xrightarrow[\psi]{} & G_1
\end{array}$$

for all $n \geq 2$, the diagram

$$\begin{array}{ccccc} G_{0,m} & \xrightarrow{\kappa^0_{n,m}} & G_{0,n} & \xrightarrow{\kappa^0_{m,n}} & G_{0,m} \\ & \searrow^{\frac{m}{n}\zeta_m} & \downarrow \zeta_n & \swarrow_{\zeta_m} & \\ & & D & & \end{array}$$

commute for all $n|m$, and making the sequence

$$G_0 \xrightarrow{\xi} X \xrightarrow{\upsilon} D \xrightarrow{\psi} G_1 \longrightarrow 0$$

a chain at D, G_1 and satisfying $\ker(\upsilon) \subseteq \overline{\xi(G_0)}$. We say a $\underline{\mathcal{K}}\mathcal{T}_u$-system is *exact* if $\mathbb{G}$ is an exact Λ-system and the sequence is exact at D, G_1 as well as $\ker(\upsilon) = \overline{\xi(G_0)}$. A $\underline{\mathcal{K}}\mathcal{T}_u$*-morphism* is a triple

$$(\underline{\alpha}, f, \beta) : (\mathbb{G}, g, X, D, \xi, \upsilon, \psi, (\zeta_n)_{n\geq 2}) \to (\mathbb{H}, h, X', D', \xi', \upsilon', \psi', (\zeta'_n)_{n\geq 2}),$$

where $\underline{\alpha} = (\alpha^j, \alpha^j_n)_{j=0,1,n\geq 2}$ is a Λ-morphism such that $\alpha^0(g) = h$, $f : X \to X'$ is a positive linear map, and $\beta : D \to D'$ is a group homomorphism, further the maps satisfies that the diagrams

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\xi} & X & \xrightarrow{\upsilon} & D & \xrightarrow{\psi} & G_1 \\ \downarrow{\scriptstyle \alpha^0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha^1} \\ H_0 & \xrightarrow[\xi']{} & X' & \xrightarrow[\upsilon']{} & D' & \xrightarrow[\psi']{} & H_1 \end{array} \qquad \begin{array}{ccc} G_{0,n} & \xrightarrow{\zeta_n} & D \\ \downarrow{\scriptstyle \alpha^0_n} & & \downarrow{\scriptstyle \beta} \\ H_{0,n} & \xrightarrow[\zeta'_n]{} & D' \end{array}$$

commute for all $n \geq 2$. A $\underline{\mathcal{K}}\mathcal{T}_u$*-isomorphism* is a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism where $\underline{\alpha}$ is a Λ-isomorphism, f is an order isomorphism and β is a group isomorphism. The $\underline{\mathcal{K}}\mathcal{T}_u$-systems form a category, which we will denote by $\underline{\mathcal{K}}\mathcal{T}_u$-$\mathfrak{Sys}$

As the name of the category suggests we have already seen an example of a $\underline{\mathcal{K}}\mathcal{T}_u$-system, namely the total invariant of a unital C^*-algebra.

Example 7.12. Let A be a unital C^*-algebra, then it's total invariant $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A)$ is indeed a $\underline{\mathcal{K}}\mathcal{T}_u$-system.

We wish to show that the splits from Proposition 7.4 induces splits for our $\underline{\mathcal{K}}\mathcal{T}_u$-systems. To show this we need to use direct limits of abelian groups. Therefore we must first take a small detour see some result about direct limits of abelian groups. All results will be from the book(monograph) Abelian Groups by László Fuchs.

Theorem 7.13 ([Fuc60] 4.1)**.** *A directed system of abelian groups* $(G_i, \pi_{j,i})_{i,j\in I}$ *has a limit* $(G = \varinjlim G_i, \pi_i)_{i\in I}$*, unique up to isomorphism, moreover*

$$G = \bigcup_{i\in I} \pi_i(G_i).$$

We are always interested in morphisms between the objects we study so lets see what homomorphisms are in this case.

Definition 7.14 ([Fuc60] p. 58)**.** Let $(A_i, \pi_{j,i})_{i,j\in I}$ and $(B_i, \rho_{j,i})_{i,j\in I}$ be directed systems over the same directed set I, then a homomorphism between the directed systems

$\Phi : (A_i, \pi_{j,i})_{i,j\in I} \to (B_i, \rho_{j,i})_{i,j\in I}$ is a family of group homomorphisms $(\varphi_i)_{i\in I}$, $\varphi_i : A_i \to B_i$ such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_{j,i}} & A_j \\ {\scriptstyle \varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_j} \\ B_i & \xrightarrow[\rho_{j,i}]{} & B_j \end{array}$$

commutes for all $i \leq j$.

Using this we can always find a unique induced group homomorphism between $\varinjlim A_i$ and $\varinjlim B_i$.

Proposition 7.15 ([Fuc60] 4.5)**.** *Let $(A_i, \pi_{j,i})_{i,j\in I}$ and $(B_i, \rho_{j,i})_{i,j\in I}$ be directed systems of abelian groups over the same index set I, and $\Phi : (A_i, \pi_{i,j})_{i,j\in I} \to (B_i, \rho_{j,i})_{i,j\in I}$ be a homomorphism between them. Then there exists a unique group homomorphism $\Phi_* : \varinjlim A_i \to \varinjlim B_i$ making the diagram*

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_{j,i}} & \varinjlim A_i \\ {\scriptstyle \varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_*} \\ B_i & \xrightarrow[\rho_{j,i}]{} & \varinjlim B_i \end{array}$$

commute for all $i \in I$.

One neat property of direct limits of abelian groups, is that they preserve exact sequences.

Theorem 7.16 ([Fuc60] 4.6)**.** *Let $(A_i, \pi_{j,i})_{i,j\in I}$, $(B_i, \rho_{j,i})_{i,j\in I}$ and $(C_i, \sigma_{j,i})_{i,j\in I}$ be directed systems of abelian groups over the same index set I and let $\Phi : (A_i, \varphi_i^j)_{i,j\in I} \to (B_i, \rho_{j,i})_{i,j\in I}$ and $\Psi : (B_i, \rho_{j,i})_{i,j\in I} \to (C_i, \sigma_{j,i})_{i,j\in I}$ be homomorphisms of directed sets such that*

$$0 \longrightarrow A_i \xrightarrow{\varphi_i} B_i \xrightarrow{\psi_i} C_i \longrightarrow 0$$

is exact for all $i \in I$. Then the sequence

$$0 \longrightarrow \varinjlim A_i \xrightarrow{\Phi_*} \varinjlim B_i \xrightarrow{\Psi_*} \varinjlim C_i \longrightarrow 0$$

is exact.

Coming back from our small detour we are now able to find our induced splittings in the $\underline{\mathcal{K}}\mathcal{T}_u$ setting.

Theorem 7.17. *Let $(\mathbb{G}, g, X, D, \xi, \upsilon, \psi, (\zeta_n)_{n\geq 2})$ be an exact $\underline{\mathcal{K}}\mathcal{T}_u$-system, then there exists (unnatural) splittings $s_D : G_1 \to D$ and $\sigma_D : D \to X/\overline{\xi(G_0)}$ making the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0/nG_0 & \underset{\sigma_n^0}{\overset{\overline{\mu}_n^0}{\rightleftarrows}} & G_{0,n} & \underset{s_n^0}{\overset{\overline{\nu}_n^0}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\ & & {\scriptstyle \frac{1}{n}\overline{\xi}}\big\downarrow & & {\scriptstyle \zeta_n}\big\downarrow & & \big\downarrow{\scriptstyle \iota_n^1} & & \\ 0 & \longrightarrow & X/\overline{\xi(G_0)} & \underset{\sigma_D}{\overset{\overline{\upsilon}}{\rightleftarrows}} & D & \underset{s_D}{\overset{\psi}{\rightleftarrows}} & G_1 & \longrightarrow & 0 \end{array}$$

commute.

Proof. We construct a direct limit of the top row in the diagram. Consider the natural numbers equipped with the partial order, $\preceq$, given by

$$n \preceq m \iff n \mid m,$$

and construct the directed sets $(G_0/nG_0, \times\frac{m}{n})_{m,n\in\mathbb{N}}$, $(G_{0,n}, \kappa^0_{m,n})_{m,n\in\mathbb{N}}$, and $(G_1[n], \iota^1_{m,n})_{m,n\in\mathbb{N}}$. By Theorem 7.13 these groups has a direct limit, we will denote the direct limit of $(G_{0,n}, \kappa^0_{m,n})_{m,n\in\mathbb{N}}$ by $(G, \theta_n)_{n\in\mathbb{N}}$. We compute the direct limit of $(G_0/nG_0, \times\frac{m}{n})_{m,n\in\mathbb{N}}$. Recall $G_0/nG_0 \cong G_0 \otimes \mathbb{Z}_n$ and consider the map $\mathrm{id}_{G_0} \otimes \times\frac{1}{n} : G_0 \otimes \mathbb{Z}_n \to G_0 \otimes \mathbb{Q}/\mathbb{Z}$ and observe

$$\left(\mathrm{id}_{G_0} \otimes \times\frac{1}{m}\right) \circ \left(\mathrm{id}_{G_0} \otimes \times\frac{m}{n}\right) = \mathrm{id}_{G_0} \otimes \times\frac{1}{n},$$

hence by Theorem 7.13

$$\varinjlim(G_0 \otimes \mathbb{Z}_n) = \bigcup_{n\in\mathbb{N}} \left(\mathrm{id}_{G_0} \otimes \times\frac{1}{n}\right)(G_0 \otimes \mathbb{Z}_n) = \bigcup_{n\in\mathbb{N}} G_0 \otimes \times\frac{1}{n}(\mathbb{Z}_n) = G_0 \otimes \mathbb{Q}/\mathbb{Z}.$$

We compute the direct limit of $(G_1[n], \iota^1_{m,n})_{m,n\in\mathbb{N}}$. Note $\iota_n : G_1[n] \to \mathrm{Tor}(G_1)$ satisfies

$$\iota_m \circ \iota^1_{m,n} = \iota_n,$$

again using Theorem 7.13

$$\varinjlim(G_1[n]) = \bigcup_{n\in\mathbb{N}} \iota_n(G_1[n]) = \bigcup_{n\in\mathbb{N}} G_1[n] = \mathrm{Tor}(G_1).$$

By Theorem 7.16 the sequence

$$0 \longrightarrow G_0 \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\overline{\mu}^0_*} G \xrightarrow{\overline{\nu}^0_*} \mathrm{Tor}(G_1) \longrightarrow 0$$

is exact. We construct a right split of this sequence. Let $x \in \mathrm{Tor}(G_1)$, then there exists an $n \in \mathbb{N}$ such that $x \in G_1[n]$, define $s_G : \mathrm{Tor}(G_1) \to G$ by

$$s_G(x) = \theta_n(s^0_n(x)).$$

If there exists $m \in \mathbb{N}$ such that $mx = 0 = nx$ then $n \mid m$ or $m \mid n$ first assume that $n \mid m$, then

$$s_G(x) = \theta_n(s^0_n(x)) = \theta_m(\kappa^0_{m,n}(s^0_n(x)) = \theta_m(s^0_m(\iota_{m,n}(x))) = \theta_m(s^0_m(x)).$$

$m \mid n$ is completely analogous and so s_G is well defined. Moreover

$$\overline{\nu}^0_* \circ s_G(x) = \overline{\nu}^0_*(\theta_n(s^0_n(x)) = \iota_n(\overline{\nu}^0_n(s^0_n(x))) = \iota_n(x) = x,$$

hence s_G is a right split and by construction it makes the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_0/nG_0 & \underset{\sigma^0_n}{\overset{\overline{\mu}^0_n}{\rightleftarrows}} & G_{0,n} & \underset{s^0_n}{\overset{\overline{\nu}^0_n}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{G_0}\otimes\times\frac{1}{n}} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \iota_n} & & \\
0 & \longrightarrow & G_0 \otimes \mathbb{Q}/\mathbb{Z} & \xrightarrow{\overline{\mu}^0_*} & G & \underset{s_G}{\overset{\overline{\nu}^0_*}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0
\end{array}$$

commute. Apply the splitting lemma 2.7 to construct a commuting left split $\sigma_G : G \to G_0 \otimes \mathbb{Q}/\mathbb{Z}$. To summarise s_G, σ_G are splittings such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_0/nG_0 & \underset{\sigma^0_n}{\overset{\overline{\mu}^0_n}{\rightleftarrows}} & G_{0,n} & \underset{s^0_n}{\overset{\overline{\nu}^0_n}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{G_0}\otimes\times\frac{1}{n}} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \iota_n} & & \\
0 & \longrightarrow & G_0 \otimes \mathbb{Q}/\mathbb{Z} & \underset{\sigma_G}{\overset{\overline{\mu}^0_*}{\rightleftarrows}} & G & \underset{s_G}{\overset{\overline{\nu}^0_*}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0,
\end{array}$$

commutes. Going back to our ζ_n maps, we have $\zeta_m \circ \kappa^0_{m,n} = \zeta_n$ applying the universal property of direct limits, yields a unique map $\zeta : G \to D$ such that $\zeta_n = \zeta \circ \theta_n$, and we expand our original diagram into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_0/nG_0 & \underset{\sigma_n^0}{\overset{\overline{\mu}_n^0}{\rightleftarrows}} & G_{0,n} & \underset{s_n^0}{\overset{\overline{\nu}_n^0}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \mathrm{id}_{G_0}\otimes\times\frac{1}{n}} & & \downarrow {\scriptstyle \theta_n} & & \downarrow {\scriptstyle \iota_n} & & \\
0 & \longrightarrow & G_0 \otimes \mathbb{Q}/\mathbb{Z} & \underset{\sigma_G}{\overset{\overline{\mu}_*^0}{\rightleftarrows}} & G & \underset{s_G}{\overset{\overline{\nu}_*^0}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \xi\otimes 1} & & \downarrow {\scriptstyle \zeta} & & \downarrow {\scriptstyle \iota_{G_1}} & & \\
0 & \longrightarrow & X/\overline{\xi(G_0)} & \xrightarrow{\upsilon} & D & \xrightarrow{\psi} & G_1 & \longrightarrow & 0,
\end{array}
$$

which commutes. Unfortunately we cannot construct our desired splittings yet, we first have to consider a new abelian group. Let $E = \psi^{-1}(\mathrm{Tor}(G_1))$, $\psi_E : E \to \mathrm{Tor}(G_1)$ and $\upsilon_E : X/\xi(G_0) \to E$ be the restriction and co-restriction to E respectively. We claim $\mathrm{Im}(\zeta) \subseteq E$, indeed by commutativity

$$\psi(\zeta(G)) = \iota_{G_1}(\overline{\nu}^0_*(G)) = \iota_{G_1}(\mathrm{Tor}(G_1)) = \mathrm{Tor}(G_1),$$

hence $\zeta(G) \subseteq \psi^{-1}(\mathrm{Tor}(G_1)) =: E$. We expand our diagram for the last time

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_0/nG_0 & \underset{\sigma_n^0}{\overset{\overline{\mu}_n^0}{\rightleftarrows}} & G_{0,n} & \underset{s_n^0}{\overset{\overline{\nu}_n^0}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \mathrm{id}_{G_0}\otimes\times\frac{1}{n}} & & \downarrow {\scriptstyle \theta_n} & & \downarrow {\scriptstyle \iota_n} & & \\
0 & \longrightarrow & G_0 \otimes \mathbb{Q}/\mathbb{Z} & \underset{\sigma_G}{\overset{\overline{\mu}_*^0}{\rightleftarrows}} & G & \underset{s_G}{\overset{\overline{\nu}_*^0}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \xi\otimes 1} & & \downarrow {\scriptstyle \zeta} & & \| & & \\
0 & \longrightarrow & X/\overline{\xi(G_0)} & \xrightarrow{\upsilon_E} & E & \xrightarrow{\psi_E} & \mathrm{Tor}(G_1) & \longrightarrow & 0 \\
 & & \| & & \downarrow {\scriptstyle \iota_D} & & \downarrow {\scriptstyle \iota_{G_1}} & & \\
0 & \longrightarrow & X/\overline{\xi(G_0)} & \xrightarrow{\upsilon} & D & \xrightarrow{\psi} & G_1 & \longrightarrow & 0.
\end{array}
$$

Note the diagram still commutes. Our strategy, will be to define splittings in the third row, and then use the splitting lemma to induce splits in the last row. Let $s_E : \mathrm{Tor}(G_1) \to E$ be given by $s_E = \zeta \circ s_G$ using commutativity

$$\psi_E \circ s_E = \psi_E \circ \zeta \circ s_G = \overline{\nu}^0_* \circ s_G = \mathrm{id}_{\mathrm{Tor}(G_1)}.$$

Applying the splitting lemma 2.7 we construct $\sigma_E : E \to X/\xi(G_0)$. As $X/\xi(G_0)$ injective, indeed it is a real vector space so by Proposition 5.15 it is divisible. Hence there exists a map $\sigma_D : D \to X/\xi(G_0)$ such that the diagram

$$
\begin{array}{ccc}
E & \overset{\iota_D}{\rightarrowtail} & D \\
\downarrow {\scriptstyle \sigma_E} & \swarrow {\scriptstyle \sigma_D} & \\
X/\overline{\xi(G_0)} & &
\end{array}
$$

commutes. We show σ_D is a left split. By commutativity of our expanded diagram

$$\sigma_D \circ \upsilon = \sigma_D \circ \iota_D \circ \upsilon_E = \sigma_E \circ \upsilon_E = \mathrm{id}_{X/\xi(G_0)}.$$

Applying the splitting lemma 2.7 for the last time, we construct $s_D : G_1 \to D$, and so the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_0/nG_0 & \underset{\sigma_n^0}{\overset{\overline{\mu}_n^0}{\rightleftarrows}} & G_{0,n} & \underset{s_n^0}{\overset{\overline{\nu}_n^0}{\rightleftarrows}} & G_1[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{G_0}\otimes\times\frac{1}{n}} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \iota_n} & & \\
0 & \longrightarrow & G_0 \otimes \mathbb{Q}/\mathbb{Z} & \underset{\sigma_G}{\overset{\overline{\mu}_*^0}{\rightleftarrows}} & G & \underset{s_G}{\overset{\overline{\nu}_*^0}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi\otimes 1} & & \downarrow{\scriptstyle \zeta} & & \| & & \\
0 & \longrightarrow & X/\overline{\xi(G_0)} & \underset{\sigma_E}{\overset{\upsilon_E}{\rightleftarrows}} & E & \underset{s_E}{\overset{\psi_E}{\rightleftarrows}} & \mathrm{Tor}(G_1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \iota_{G_1}} & & \\
0 & \longrightarrow & X/\overline{\xi(G_0)} & \underset{\sigma_D}{\overset{\upsilon}{\rightleftarrows}} & D & \underset{s_D}{\overset{\psi}{\rightleftarrows}} & G_1 & \longrightarrow & 0
\end{array}$$

commutes. □

Having found induced splittings we will do as in the Λ-system case and consider what category $\underline{\mathcal{KT}}_u$ could arise from.

Definition 7.18. A $\mathcal{KT}_u$*-system* is a quintuple (G_0, g, G_1, X, ρ) where G_0, G_1 are abelian groups, $g \in G_0$, X is an order unit space, and $\rho : G_0 \to X$ is a group homomorphism such that $\rho(g) = e$ the order unit in X. A $\mathcal{KT}_u$*-morphism* is a triple

$$(\alpha^0, \alpha^1, f) : (G_0, g, G_1, X, \rho) \to (H_0, h, H_1, X', \rho'),$$

where $\alpha^j : G_j \to H_j$ are group homomorphisms, where $\alpha^0(g) = h$, and $f : X \to X'$ is a positive linear map, such that the diagram

$$\begin{array}{ccc}
G_0 & \xrightarrow{\rho} & X \\
\downarrow{\scriptstyle \alpha^0} & & \downarrow{\scriptstyle f} \\
H_0 & \xrightarrow[\rho']{} & X'
\end{array}$$

commutes. We say a $\mathcal{KT}_u$-morphism is a $\mathcal{KT}_u$-isomorphism if α^j are both group isomorphisms and f is an order isomorphism. This forms a category which we will denote by $\mathcal{KT}_u$-$\mathfrak{sys}$.

We now see that our intuition is spot on, that is we can induce objects in $\underline{\mathcal{KT}}_u$-$\mathfrak{sys}$ from $\mathcal{KT}_u$-$\mathfrak{sys}$.

Proposition 7.19. *Let* (G_0, g, G_1, X, ρ) *be a* $\mathcal{KT}_u$*-system. Then* $(\Lambda(G_0, G_1), g, X, X/\overline{\rho(G_0)} \oplus G_1, \rho, \upsilon, \psi, (\zeta_n)_{n\geq 2})$ *where* $\upsilon : X \to X/\overline{\xi(G_0)} \oplus G_1$ *is given by* $\upsilon(x) = ([x], 0)$*,* $\psi : X/\overline{\xi(G_0)} \oplus G_1 \to G_1$ *is given by* $\psi([x], y) = y$ *and* $\zeta_n : G_0/nG_0 \oplus G_1[n] \to X/\overline{\xi(G_0)} \oplus G_1$ *is given by* $\zeta_n([x], y) = \left(\left[\frac{1}{n}\rho(x)\right], \iota_n^1(y)\right)$ *is an exact* $\underline{\mathcal{KT}}_u$*-system. We call this system the* trivial system *and denote it by* $\mathrm{T}(G_0, g, G_1, X, \rho)$.

Proof. We showing the diagram

$$\begin{array}{ccccc}
G_0 & \xrightarrow{\mu_n^0} & G_0/nG_0 \oplus G_1[n] & \xrightarrow{\nu_n^0} & G_1 \\
\downarrow{\scriptstyle \frac{1}{n}\rho} & & \downarrow{\scriptstyle \zeta_n} & & \| \\
X & \xrightarrow[\upsilon]{} & X/\overline{\rho(G_0)} \oplus G_1 & \xrightarrow{\psi} & G_1,
\end{array}$$

commutes. First the left square:

$$\zeta_n(\mu_n^0(x)) = \zeta_n([x], 0) = \left(\left[\frac{1}{n}\rho(x)\right], 0\right) = \upsilon\left(\frac{1}{n}\rho(x)\right).$$

Now the right square:

$$\psi(\zeta_n([x], y)) = \psi\left(\left[\frac{1}{n}\rho(x)\right], \iota_n^1(y)\right) = \iota_n^1(y) = \nu_n^0(y).$$

We show the diagram

$$\begin{array}{ccccc} G_0/mG_0 \oplus G_1[m] & \xrightarrow{\kappa^0_{n,m}} & G_0/nG_0 \oplus G_1[n] & \xrightarrow{\kappa^0_{m,n}} & G_0/mG_0 \oplus G_1[m] \\ & \searrow^{\frac{m}{n}\zeta_m} & \downarrow \zeta_n & \swarrow^{\zeta_m} & \\ & & X/\overline{\rho(G_0)} \oplus G_1 & & \end{array}$$

commutes. As is tradition by now we start with the left triangle:

$$\zeta_n(\kappa^0_{n,m}([x], y)) = \zeta_n\left([x], \frac{m}{n}y\right) = \left(\left[\frac{1}{n}\rho(x)\right], \frac{m}{n}y\right) = \frac{m}{n}\left(\left[\frac{1}{m}\rho(x)\right], y\right) = \frac{m}{n}\zeta_m([x], y).$$

The right triangle:

$$\zeta_m(\kappa^0_{m,n}([x], y)) = \zeta_m\left(\left[\frac{m}{n}x\right], y\right) = \left(\left[\frac{1}{m}\rho\left(\frac{m}{n}x\right)\right], y\right) = \left(\left[\frac{1}{n}\rho(x)\right], y\right) = \zeta_n([x], y).$$

Last but not least we show that

$$G_0 \xrightarrow{\xi} X \xrightarrow{\upsilon} X/\overline{\xi(G_0)} \oplus G_1 \xrightarrow{\psi} G_1 \longrightarrow 0$$

is exact at G_1 and $X/\overline{\xi(G_0)} \oplus G_1$ as well as $\ker(\upsilon) = \overline{\xi(G_0)}$. As ψ is the projection on the second coordinate, it is surjective. Since $\upsilon(X) = (X/\overline{\xi(G_0)}, 0)$ we get exactness at $X/\overline{\xi(G_0)} \oplus G_1$ and $\ker(\upsilon)$ is exactly $\overline{\xi(G_0)}$. □

We are now able to proof our Bödigheimer-esque result for $\underline{\mathcal{K}}\mathcal{T}_u$-systems.

Theorem 7.20. *Let* $\mathbb{K} = (\mathbb{G}, g, X, D, \xi, \upsilon, \psi, (\zeta_n)_{n\geq 2})$ *be an exact* $\underline{\mathcal{K}}\mathcal{T}_u$*-system. Then* $\mathbb{K}$ *is unnaturally isomorphic to the trivial system* $T(F_\Lambda(\mathbb{G}), g, X, \xi)$*.*

Proof. The proof strategy will be to construct a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism and then apply the 5-lemma to see it is indeed an isomorphism. Recall $\Lambda(F_\Lambda(\mathbb{G})) \cong \mathbb{G}$ by Bödigheimer call this isomorphism $\underline{\alpha}$, pick f to be the identity of X, and let $\beta : X/\overline{\xi(G_0)} \oplus G_1 \to D$ be given by

$$\beta([x], y) = \upsilon(x) + s_D(y),$$

where $s_D : G_1 \to D$ is the induced split from Theorem 7.17. We show $(\underline{\alpha}, \mathrm{id}_X, \beta)$ is a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. As $\alpha^0 = \mathrm{id}_{G_0}$ we have $\alpha^0(g) = g$. We show the diagram

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\xi} & X & \xrightarrow{\upsilon'} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow{\psi'} & G_1 \\ \| & & \| & & \downarrow \beta & & \| \\ G_0 & \xrightarrow[\xi]{} & X & \xrightarrow[\upsilon]{} & D & \xrightarrow[\psi]{} & G_1 \end{array}$$

commutes. Trivially the left square commutes, we now show the central square:

$$\beta(\upsilon'(x)) = \beta([x], 0) = \upsilon(x).$$

The right square:

$$\psi(\beta([x], y)) = \underbrace{\psi(\upsilon(x))}_{=0} + \underbrace{\psi(s_D(y))}_{=\mathrm{id}_{G_1}} = y.$$

We show

$$\begin{array}{ccc} G_0/nG_0 \oplus G_1[n] & \xrightarrow{\zeta_n'} & X/\overline{\xi(G_0)} \oplus G_1 \\ {\scriptstyle \alpha_n^0}\downarrow & & \downarrow{\scriptstyle \beta} \\ G_{0,n} & \xrightarrow[\zeta_n]{} & D, \end{array}$$

commutes. First observe

$$\beta(\zeta_n'([x], y)) = \beta\left(\left[\frac{1}{n}\xi(x)\right], \iota_n^1(y)\right) = \upsilon\left(\frac{1}{n}\xi(x)\right) + s_D(\iota_n^1(y)).$$

By Theorem 7.17

$$\upsilon\left(\frac{1}{n}\xi(x)\right) + s_D(\iota_n^1(y)) = \zeta_n(\mu_n^0(x)) + \zeta_n(s_n^0(y)) = \zeta_n(\alpha_n^0([x], y)).$$

As $\underline{\alpha}$ was chosen to be a Λ-isomorphism and id_X is an order isomorphism, we need only show that β is a group isomorphism. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X/\overline{\xi(G_0)} & \xrightarrow{\overline{\upsilon}'} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow{\psi'} & G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & X/\overline{\xi(G_0)} & \xrightarrow[\overline{\upsilon}]{} & D & \xrightarrow[\psi]{} & G_1 & \longrightarrow & 0 \end{array}$$

Note β makes the diagram commute, as $\overline{\upsilon}'([x]) = ([x], 0)$, and by the 5-lemma β is a group isomorphism. □

As in the Λ-system case we consider the forgetful functor and it's relation to trivial systems.

Definition 7.21. Let $\mathbb{K}$ be a $\underline{\mathcal{K}}\mathcal{T}_u$-system and let $(\underline{\alpha}, \gamma, \beta)$ be a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism, the forgetful functor $F_{\underline{\mathcal{K}}\mathcal{T}_u} : \underline{\mathcal{K}}\mathcal{T}_u\text{-}\mathfrak{sys} \to \mathcal{K}\mathcal{T}_u\text{-}\mathfrak{sys}$ maps objects $\mathbb{K} = (\mathbb{G}, g, X, D, \xi, \upsilon, \psi, (\zeta_n)_{n\geq 2})$ to $F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}) = (G_0, g, G_1, X, \xi)$ and morphisms $(\underline{\alpha}, \gamma, \beta)$ to $F_{\underline{\mathcal{K}}\mathcal{T}_u}(\underline{\alpha}, \gamma, \beta) = (\alpha^0, \alpha^1, \gamma)$.

Using this definition we can lift morphisms from $\mathcal{K}\mathcal{T}_u\text{-}\mathfrak{sys}$ to morphisms in $\underline{\mathcal{K}}\mathcal{T}_u\text{-}\mathfrak{sys}$.

Theorem 7.22. *Let*

$$\begin{aligned} \mathbb{K} &= (\mathbb{G}, g, X, D, \xi_K, \upsilon, \psi, (\zeta_n)_{n\geq 2}), \\ \mathbb{L} &= (\mathbb{H}, h, Y, E, \xi_L, \upsilon', \psi', (\zeta_n')_{n\geq 2}) \end{aligned}$$

be exact $\underline{\mathcal{K}}\mathcal{T}_u$-systems. Then a homomorphism $\varphi : F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}) \to F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{L})$ lifts to a homomorphism $\Phi : \mathbb{K} \to \mathbb{L}$ further if φ is an isomorphism so is Φ, and $F_{\underline{\mathcal{K}}\mathcal{T}_u}(\Phi) = \mathrm{id}_{\mathbb{L}}$.

Proof. Let $(\alpha^0, \alpha^1, f) : F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}) \to F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{L})$ be a $\mathcal{K}\mathcal{T}_u$-morphism, and

$$(\Lambda(G_0, G_1), g, X, \xi_K, X/\overline{\xi(G_0)} \oplus G_1, \upsilon_K, \psi_K, (\zeta_n^K)_{n\geq 2}),$$
$$(\Lambda(H_0, H_1), h, Y, \xi_L, Y/\overline{\xi(H_0)} \oplus H_1, \upsilon_L, \psi_L, (\zeta_n^L)_{n\geq 2}),$$

be the induced objects from theorem 7.20. Let $\Lambda(\alpha^0, \alpha^1)$ be the induced Λ-morphism, and define $\beta : X/\overline{\xi_K(G_0)} \oplus G_1 \to Y/\overline{\xi_L(H_0)} \oplus H_1$ by

$$\beta = \overline{\upsilon}_L \circ \overline{f} \circ \sigma_K + s_L \circ \alpha^1 \circ \psi_K,$$

where $s_L : H_1 \to Y/\overline{\xi(H_0)} \oplus H_1$ is the induced splitting, and $\overline{\upsilon}, \overline{f}$ are the induced quotient maps. We show $\Phi = (\Lambda(\alpha^0, \alpha^1), f, \beta)$ is a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. We consider the diagram

$$\begin{array}{ccccccc}
G_0 & \xrightarrow{\xi_K} & X & \xrightarrow{\upsilon_K} & X/\overline{\xi_K(G_0)} \oplus G_1 & \xrightarrow{\psi_K} & G_1 \\
{\scriptstyle\alpha^0}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha_1}\downarrow \\
H_0 & \xrightarrow{\xi_L} & Y & \xrightarrow{\upsilon_L} & Y/\overline{\xi(H_0)} \oplus H_1, & \xrightarrow{\psi_L} & H_1
\end{array}$$

and show it commutes. The left square commutes as we come from a $\mathcal{K}\mathcal{T}_u$-system, for the central square:

$$\beta(\upsilon_K(x)) = \beta([x], 0) = \overline{\upsilon}_L(\overline{f}(\sigma_K([x], 0))) = \overline{\upsilon}_L([f(x)]) = \upsilon_L(f(x)).$$

The right square commutes as

$$\psi_L \circ \beta = \underbrace{\psi_L \circ \overline{\upsilon}_L}_{=0} \circ \overline{f} \circ \sigma_K + \underbrace{\psi_L \circ s_L}_{=\mathrm{id}_{H_1}} \circ \alpha^1 \circ \psi_K = \alpha^1 \circ \psi_K.$$

We show

$$\begin{array}{ccc}
G_0/nG_0 \oplus G_1[n] & \xrightarrow{\zeta_n^K} & X/\overline{\xi_K(G_0)} \oplus G_1 \\
{\scriptstyle\alpha_n^0}\downarrow & & {\scriptstyle\beta}\downarrow \\
H_0/nH_0 \oplus H_1[n] & \xrightarrow{\zeta_n^L} & Y/\overline{\xi(H_0)} \oplus H_1,
\end{array}$$

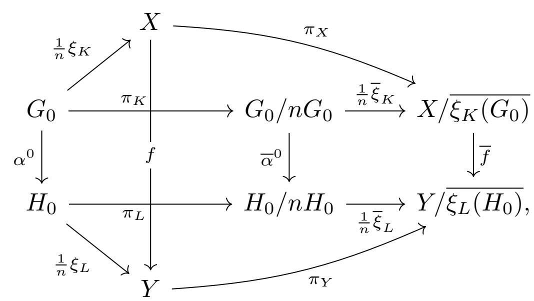
commutes. Recall $\alpha_n^0 = \overline{\mu}_{0,n}^L \circ \overline{\alpha}^0 \circ \sigma_{0,n}^K + s_{0,n}^L \circ \widetilde{\alpha}^1 \circ \overline{\nu}_{0,n}^K$ and consider $\zeta_n^L \circ \overline{\mu}_{0,n}^L \circ \overline{\alpha}^0 \circ \sigma_{0,n}^K$. For the diagram

$$\begin{array}{ccccc}
H_0 & \xrightarrow{\pi_{0,n}^L} & H_0/nH_0 & \xrightarrow{\overline{\mu}_{0,n}^L} & H_0/nH_0 \oplus H_1[n] \\
{\scriptstyle\frac{1}{n}\xi_L}\downarrow & & {\scriptstyle\frac{1}{n}\overline{\xi}_L}\downarrow & & \downarrow{\scriptstyle\zeta_n^L} \\
Y & \xrightarrow[\pi_Y]{} & Y/\overline{\xi(H_0)} & \rightarrowtail_{\overline{\upsilon}} & Y/\overline{\xi(H_0)} \oplus H_1
\end{array}$$

the left square commutes, using this

$$\zeta_n^L \circ \overline{\mu}_{0,n}^L \circ \pi_{0,n}^L = \zeta_n^L \circ \mu_{0,n}^L = \upsilon_L \circ \frac{1}{n}\xi_L = \overline{\upsilon}_L \circ \pi_Y \circ \frac{1}{n}\xi_L = \overline{\upsilon} \circ \frac{1}{n}\overline{\xi}_L \circ \pi_L.$$

Since $\pi_{0,n}^L$ is surjective the right square commutes. We consider the diagram



where all squares but

$$\begin{array}{ccc} G_0/nG_0 & \xrightarrow{\frac{1}{n}\overline{\xi}_K} & X/\overline{\xi_K(G_0)} \\ {\scriptstyle \overline{\alpha}_0}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} \\ H_0/nH_0 & \xrightarrow[\frac{1}{n}\overline{\xi}_L]{} & Y/\overline{\xi_L(H_0)} \end{array}$$

commutes, either by construction or as we have a $\mathcal{KT}_u$-morphism. We show the last square commutes:

$$\begin{aligned} \overline{f} \circ \frac{1}{n}\overline{\xi}_K \circ \pi_K &= \overline{f} \circ \pi_X \circ \frac{1}{n}\xi_K \\ &= \pi_Y \circ f \circ \frac{1}{n}\xi_K \\ &= \pi_Y \circ \frac{1}{n}\xi_L \circ \alpha^0 \\ &= \frac{1}{n}\overline{\xi}_L \circ \pi_L \circ \alpha_0 \\ &= \frac{1}{n}\overline{\xi}_L \circ \overline{\alpha}^0 \circ \pi_K. \end{aligned}$$

Lastly consider

$$\begin{array}{ccc} G_0/nG_0 \oplus G_1[n] & \xrightarrow{\sigma_{0,n}^K} & G_0/nG_0 \\ {\scriptstyle \zeta_n^K}\big\downarrow & & \big\downarrow{\scriptstyle \frac{1}{n}\overline{\xi}_K} \\ X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow[\sigma_K]{} & X/\overline{\xi(G_0)} \end{array}$$

which commutes by Theorem 7.17. Combining the above

$$\zeta_n^L \circ \overline{\mu}_{0,n}^L \circ \overline{\alpha}^0 \circ \sigma_{0,n}^K = \overline{v} \circ \frac{1}{n}\overline{\xi}_L \circ \overline{\alpha}^0 \circ \sigma_{0,n}^K = \overline{v} \circ \overline{f} \circ \frac{1}{n}\overline{\xi}_K \circ \sigma_{0,n}^K = \overline{v} \circ \overline{f} \circ \sigma_K \circ \zeta_n^K.$$

We now focus on $\zeta_n^L \circ s_{0,n}^L \circ \widetilde{\alpha}^1 \circ \overline{\nu}_{0,n}^K$. By Theorem 7.17

$$\zeta_n^L \circ s_{0,n}^L \circ \widetilde{\alpha}^1 \circ \overline{\nu}_{0,n}^K = s_L \circ \iota_{1,n}^L \circ \widetilde{\alpha}^1 \circ \overline{\nu}_{0,n}^K,$$

and by construction of $\widetilde{\alpha}^1$ we get $s_L \circ \iota_{1,n}^L \circ \widetilde{\alpha}^1 \circ \overline{\nu}_{0,n}^K = s_L \circ \alpha^1 \circ \iota_{1,n}^K \circ \overline{\nu}_{0,n}^K$. Lastly as the trivial system is a $\underline{\mathcal{K}}\mathcal{T}_u$-system

$$s_L \circ \alpha^1 \circ \iota_{1,n}^K \circ \overline{\nu}_{0,n}^K = s_L \circ \alpha^1 \circ \nu_{0,n}^K = s_L \circ \alpha^1 \circ \psi_K \circ \zeta_n^K.$$

Combining this with our previous case

$$\zeta_n^L \circ \alpha_n^0 = \overline{\nu}_L \circ \overline{f} \circ \sigma_K \circ \zeta_n^K + s_L \circ \alpha^1 \circ \psi_K \circ \zeta_n^K = \beta \circ \zeta_n^K.$$

By construction of Φ, $F_{\underline{\mathcal{K}}\mathcal{T}_u}(\Phi) = \mathrm{id}_{\mathbb{L}}$, what is left to show is that isomorphisms lift to isomorphisms. If (α^0, α^1, f) is an isomorphism then f is an order isomorphism and both α^j's are group isomorphisms, which means that $\Lambda(\alpha^0, \alpha^1)$ is a Λ-isomorphism by Proposition 2.6. Hence we need only show β is an isomorphism, we will do this by computing the explicit inverse.
Let $\beta^{-1} : Y/\overline{\xi_L(H_0)} \oplus H_1 \to X/\overline{\xi_K(G_0)} \oplus G_1$ be given by

$$\beta^{-1} = \overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \sigma_L + s_K \circ \left(\alpha^1\right)^{-1} \circ \psi_L.$$

Using the obvious notational changes the above computations show $(\Lambda((\alpha^0)^{-1}, (\alpha^1)^{-1}), f^{-1}, \beta^{-1})$ is a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. We show it is the inverse. As is tradition we show left inverse first:

$$\begin{aligned}
\beta^{-1} \circ \beta &= (\overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \sigma_L + s_K \circ \left(\alpha^1\right)^{-1} \circ \psi_L) \circ (\overline{\upsilon}_L \circ \overline{f} \circ \sigma_K + s_L \circ \alpha^1 \circ \psi_K) \\
&= \overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \sigma_L \circ \overline{\upsilon}_L \circ \overline{f} \circ \sigma_K + \overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \underbrace{\sigma_L \circ s_L}_{=0} \circ \alpha^1 \circ \psi_K \\
&\quad + s_K \circ \left(\alpha^1\right)^{-1} \circ \underbrace{\psi_L \circ \overline{\upsilon}_L}_{=0} \circ \overline{f} \circ \sigma_K + s_K \circ (\alpha^1)^{-1} \circ \psi_L \circ s_L \circ \alpha^1 \circ \psi_K \\
&= \overline{\upsilon}_K \circ \sigma_K + s_K \circ \psi_K \\
&= \mathrm{id}_{X/\overline{\xi_K(G_0)} \oplus G_1} - s_K \circ \psi_K + s_K \circ \psi_K \\
&= \mathrm{id}_{X/\overline{\xi_K(G_0)} \oplus G_1},
\end{aligned}$$

where $\sigma_- \circ s_- = 0$ as s_- comes from the splitting lemma. We show the right inverse:

$$\begin{aligned}
\beta \circ \beta^{-1} &= (\overline{\upsilon}_L \circ \overline{f} \circ \sigma_K + s_L \circ \alpha^1 \circ \psi_K) \circ (\overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \sigma_L + s_K \circ \left(\alpha^1\right)^{-1} \circ \psi_L) \\
&= \overline{\upsilon}_L \circ \overline{f} \circ \sigma_K \circ \overline{\upsilon}_K \circ (\overline{f})^{-1} \circ \sigma_L + \overline{\upsilon}_L \circ \overline{f} \circ \underbrace{\sigma_K \circ s_K}_{=0} \circ (\alpha^1)^{-1} \circ \psi_L \\
&\quad + s_L \circ \alpha^1 \circ \underbrace{\psi_K \circ \overline{\upsilon}_K}_{=0} \circ (\overline{f})^{-1} \circ \sigma_L + s_L \circ \alpha^1 \circ \psi_K \circ s_K \circ (\alpha^1)^{-1} \circ \psi_L \\
&= \overline{\upsilon}_L \circ \sigma_L + s_L \circ \psi_L \\
&= \mathrm{id}_{Y/\overline{\xi_L(H_0)} \oplus H_1} - s_L \circ \psi_L + s_L \circ \psi_L \\
&= \mathrm{id}_{Y/\overline{\xi_L(H_0)} \oplus H_1}.
\end{aligned}$$

□

We observe this makes T into a functor.

Definition 7.23. Let $\mathrm{T} : \mathcal{K}\mathcal{T}_u\text{-}\mathfrak{sys} \to \underline{\mathcal{K}}\mathcal{T}_u\text{-}\mathfrak{sys}$, be the functor mapping objects to their induced trivial object, and morphisms to their lifted morphisms as in 7.22, we will denote this functor as the *trivialisation functor*.

We are now almost 60 pages into this thesis and we are finally able to see why the title is Automorphisms on classifiable C^*-algebras.

Corollary 7.24. *Let $\mathbb{K}$ be an exact $\underline{\mathcal{K}}\mathcal{T}_u$-system and denote by $\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u} : \mathrm{Aut}(\mathbb{K}) \to \mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}))$ the induced map of the forgetful functor. Then there exists an unnatural group homomorphism*

$$\Phi : \mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})) \to \mathrm{Aut}(\mathbb{K}),$$

such that $\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u} \circ \Phi = \mathrm{id}_{\mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}))}$.

Proof. By Theorem 7.20, $\mathbb{K} \cong \mathrm{T}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}))$, call this isomorphism Ψ, further by construction $F_{\underline{\mathcal{K}}\mathcal{T}_u}(\Psi) = \mathrm{id}_{F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})}$. Induce a group isomorphism $\widetilde{\Psi} : \mathrm{Aut}(\mathbb{K}) \to \mathrm{Aut}(\mathrm{T}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})))$ given by $\widetilde{\Psi}(\varphi) = \Psi \circ \varphi \circ \Psi^{-1}$. Further use the trivialisation functor to induce a group homomorphism $\widetilde{\Phi} : \mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})) \to \mathrm{Aut}(\mathrm{T}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})))$ by $\widetilde{\Phi}(\varphi) = \mathrm{T}(\varphi)$. Define $\Phi : \mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K})) \to \mathrm{Aut}(\mathbb{K})$ by $\Phi = \widetilde{\Psi}^{-1} \circ \widetilde{\Phi}$. Note Φ is a group homomorphism as both constituents are, hence to show Φ is well defined we need $\Phi(\varphi) \in \mathrm{Aut}(\mathbb{K})$. We calculate $\Phi(\varphi)$, let $\varphi = (\alpha^0, \alpha^1, \gamma) \in \mathrm{Aut}(F_{\underline{\mathcal{K}}\mathcal{T}_u}(\mathbb{K}))$.

$$\begin{aligned}\Phi(\varphi) &= \widetilde{\Psi}^{-1} \circ \widetilde{\Phi}(\varphi) \\ &= \widetilde{\Psi}^{-1}(\Lambda(\alpha^0, \alpha^1), \gamma, \beta_\Phi) \\ &= (\Lambda(\mathrm{id}_{G_0}, \mathrm{id}_{G_1})^{-1}, \mathrm{id}_X, \beta_\Psi^{-1}) \circ (\Lambda(\alpha^0, \alpha^1), \gamma, \beta_\Phi) \circ (\Lambda(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}), \mathrm{id}_X, \beta_\Psi) \\ &= (\Lambda(\alpha^0, \alpha^1), \gamma, \beta_\Psi^{-1} \circ \beta_\Phi \circ \beta_\Psi).\end{aligned}$$

We show $(\Lambda(\alpha^0, \alpha^1), \gamma, \beta_\Psi^{-1} \circ \beta_\Phi \circ \beta_\Psi)$ is a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. As φ is a $\mathcal{K}\mathcal{T}_u$-morphism $\alpha^0(g) = g$. We show the diagram

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\xi} & X & \xrightarrow{\upsilon} & D & \xrightarrow{\psi} & G_1 \\ \downarrow{\scriptstyle \alpha^0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta_\Psi^{-1}\circ\beta_\Phi\circ\beta_\Psi} & & \downarrow{\scriptstyle \alpha^1} \\ G_0 & \xrightarrow[\xi]{} & X & \xrightarrow[\upsilon]{} & D & \xrightarrow[\psi]{} & G_1, \end{array}$$

commutes. The left square commutes, as we come from a $\mathcal{K}\mathcal{T}_u$-morphism, we now expand the central and right squares:

$$\begin{array}{ccccc} X & \xrightarrow{\upsilon} & D & \xrightarrow{\psi} & G_1 \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \beta_\Psi} & & \downarrow{\scriptstyle \mathrm{id}_{G_1}} \\ X & \xrightarrow{\upsilon'} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow{\psi'} & G_1 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \beta_\Phi} & & \downarrow{\scriptstyle \alpha^1} \\ X & \xrightarrow{\upsilon'} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow{\psi'} & G_1 \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \beta_\Psi^{-1}} & & \downarrow{\scriptstyle \mathrm{id}_{G_1}} \\ X & \xrightarrow[\upsilon]{} & D & \xrightarrow[\psi]{} & G_1. \end{array}$$

We observe all squares commute, as each comes from a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. We show

$$\begin{array}{ccc} G_{0,n} & \xrightarrow{\alpha_n^0} & G_{0,n} \\ {\scriptstyle \zeta_n}\downarrow & & \downarrow{\scriptstyle \zeta_n} \\ D & \xrightarrow[\beta_\Psi^{-1}\circ\beta_\Phi\circ\beta_\Psi]{} & D, \end{array}$$

commutes. We expand our diagram

$$\begin{array}{ccccccc} G_{0,n} & \xrightarrow{\alpha^0_{\Psi,n}} & G_0/nG_0 \oplus G_1[n] & \xrightarrow{\alpha^0_{\Phi,n}} & G_0/nG_0 \oplus G_1[n] & \xrightarrow{(\alpha^0_{\Psi,n})^{-1}} & G_{0,n} \\ \zeta_n \downarrow & & \downarrow \zeta'_n & & \downarrow \zeta'_n & & \downarrow \zeta_n \\ D & \xrightarrow[\beta_\Psi]{} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow[\beta_\Phi]{} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow[\beta_\Psi^{-1}]{} & D. \end{array}$$

Again all the squares commute as their respective maps come from $\underline{\mathcal{K}}\mathcal{T}_u$-morphisms, so Φ is well defined. Lastly observe

$$(\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u} \circ \Phi)(\varphi) = \widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u}(\Lambda(\alpha^0, \alpha^1), \gamma, \beta_\Psi^{-1} \circ \beta_\Phi \circ \beta_\Psi) = (\alpha^0, \alpha^1, \gamma).$$

□

With this result we will move away from abstract nonsense and return to C^*-algebra land albeit only for a little bit.

8 Lifting Group Actions

In this section we will consider automorphism groups of C^*-algebras and show for a subset of particularly nice C^*-algebras that we can lift group actions on KT_u all the way up to group actions on the automorphism group modulo approximate inner automorphisms, $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}$. Lets first define one of the words we just introduced.

Definition 8.1. Let A be a C^*-algebra, $\mathrm{Aut}(A)$ be it's automorphism group, and let $\varphi \in \mathrm{Aut}(A)$ be an automorphism. We say that φ is *inner* if there exists a unitary, u, in the unitisation of A, $\widetilde{A}$, such that $\varphi(a) = uau^*$ for all $a \in A$. We say φ is *approximately inner* if for every finite subset, F, of A, and for every $\varepsilon > 0$ there is an inner automorphism, ψ such that $||\varphi(a) - \psi(a)|| < \varepsilon$, $a \in F$. Denote the normal subgroup of $\mathrm{Aut}(A)$ consisting of approximately inner automorphisms by $\overline{\mathrm{Inn}}(A)$.

To show one of our main results of this section we need a corollary from [CGS+23], namely corollary 9.10.

Theorem 8.2 ([CGS+23], corollary 9.10)**.** *Let A be a unital simple separable nuclear $\mathcal{Z}$-stable C^*-algebra satisfying the universal coefficient theorem. Then the quotient* $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A)$ *is isomorphic to the automorphism group of* $\underline{\mathrm{K}}\mathrm{T_u}(A)$*,* $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A) \cong \mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$*, implemented by* $\underline{\mathrm{K}}\mathrm{T_u}$.

With this we are ready to prove our result, it should be noted that Christopher Schafhauser and a Ph.D. student has a proof of this as well, however as far as the author knows the proof uses different techniques.

Corollary 8.3. *Let A be a unital C^*-algebra, and G be a discrete group acting on $KT_u(A)$. Then this action lifts (non-canonically) to an action on* $\underline{\mathrm{K}}\mathrm{T_u}(A)$*. In particular if A is also simple separable nuclear $\mathcal{Z}$-stable and satisfy the universal coefficient theorem, then every group action of G on $KT_u(A)$ lifts to a homomorphism $G \to \mathrm{Aut}(G)/\overline{\mathrm{Inn}}(A)$.*

Proof. By definition a group action on $\mathrm{KT_u}(A)$ is a group homomorphism $\chi : G \to \mathrm{Aut}(\mathrm{KT_u}(A))$, by Corollary 7.24 we can find a group homomorphism $\Phi : \mathrm{Aut}(\mathrm{KT_u}(A)) \to \mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$ such that $F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \Phi = \mathrm{id}_{\mathrm{Aut}(\mathrm{KT_u}(A))}$. Define $\widetilde{\chi} : G \to \mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$ by $\widetilde{\chi} = \Phi \circ \chi$, then $F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \widetilde{\chi} = F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \Phi \circ \chi = \chi$ and thus we have shown the first part. Now assume A is also simple, separable, nuclear, $\mathcal{Z}$-stable, and satisfy the universal coefficient theorem. By Theorem 8.2 $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A) \cong \mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$, implemented by $\underline{\mathrm{K}}\mathrm{T_u}$, we define $\overline{\chi} : G \to \mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A)$ by $\overline{\chi} = \underline{\mathrm{K}}\mathrm{T_u}^{-1} \circ \widetilde{\chi}$. Recall $\mathrm{KT_u} = F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \underline{\mathrm{K}}\mathrm{T_u}$,

$$\mathrm{KT_u} \circ \overline{\chi} = F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \underline{\mathrm{K}}\mathrm{T_u} \circ \underline{\mathrm{K}}\mathrm{T_u}^{-1} \circ \widetilde{\chi} = F_{\underline{\mathrm{K}}\mathrm{T_u}} \circ \widetilde{\chi} = \chi.$$

□

Let us calculate an example of what we have just seen.

Example 8.4. Let A be a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebra satisfying the universal coefficient theorem, such that $KT_u(A) = (\mathbb{Z}, 1, \mathbb{Z}_2, \mathbb{R}, \rho)$, where ρ is the canonical inclusion of $\mathbb{Z}$ into $\mathbb{R}$. We calculate $\mathrm{Aut}(KT_u(A))$, recall that $\mathbb{Z}$ has two group automorphisms, $\times 1$ and $\times(-1)$, $\mathbb{Z}_2$ only has one group automorphism $\times 1$, and for $\mathbb{R}$ consider any positive linear map. For a KT_u-morphism the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xhookrightarrow{\rho} & \mathbb{R} \\ {\scriptstyle \alpha^0}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathbb{Z} & \xhookrightarrow[\rho]{} & \mathbb{R} \end{array}$$

must commute, this forces α^0 to be $\times 1$ and γ to be unital, the only unital positive linear automorphism of $\mathbb{R}$ is $\times 1$. Thus we have one KT_u automorphism, namely $(\times 1, \times 1, \times 1)$. We compute $\mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$, that is, we compute the automorphisms for the induced trivial system

$$\underline{\mathrm{K}}\mathrm{T_u}(A) = (\Lambda(\mathbb{Z}, \mathbb{Z}_2), 1, \mathbb{R}, \mathbb{T} \oplus \mathbb{Z}_2, \rho, \iota_1, \pi_2, (\zeta)_{n \geq 2}),$$

where $\iota_1 : \mathbb{R} \to \mathbb{T} \oplus \mathbb{Z}_2$ is the inclusion, $\pi_2 : \mathbb{T} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ is the projection, and $\zeta_n : \mathbb{Z}_n \oplus \mathbb{Z}_2[n] \to \mathbb{T} \oplus \mathbb{Z}_2$ is given by $\zeta_n([x]_n, y) = ([\frac{1}{n}x]_{\mathbb{R}/\mathbb{Z}}, \iota_n(y))$. Before computing the automorphisms of $\underline{\mathrm{K}}\mathrm{T_u}(A)$ we look a bit closer at the Λ-system $\Lambda(\mathbb{Z}, \mathbb{Z}_2)$. Observe

$$\mathbb{Z}_2[n] = \begin{cases} \mathbb{Z}_2 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd,} \end{cases}$$

and

$$\mathbb{Z}_2 \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(2,n)} = \begin{cases} \mathbb{Z}_2 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd.} \end{cases}$$

We compute $\mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$, by the above we have different cases depending on if $n, m \geq 2$ are even, odd or a combination. For the rest of this example we will assume that $n|m$. The first case we tackle is when both n, m are even. We first compute the automorphisms of $\Lambda(\mathbb{Z}, \mathbb{Z}_2)$. Thus the 6 diagrams must all commute

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\mathrm{id}_{\mathbb{Z}}} & \mathbb{Z} \\ {\scriptstyle \iota_1^n}\downarrow & & \downarrow{\scriptstyle \iota_1^n} \\ \mathbb{Z}_n \oplus \mathbb{Z}_2 & \xrightarrow[\alpha_n^0]{} & \mathbb{Z}_n \oplus \mathbb{Z}_2 \end{array} \qquad \begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{\mathrm{id}_{\mathbb{Z}_2}} & \mathbb{Z}_2 \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}_2}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}_2}} \\ \mathbb{Z}_2 & \xrightarrow[\alpha_n^1]{} & \mathbb{Z}_2 \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}_n \oplus \mathbb{Z}_2 & \xrightarrow{\alpha_n^0} & \mathbb{Z}_n \oplus \mathbb{Z}_2 \\ {\scriptstyle \pi_2^0}\downarrow & & \downarrow{\scriptstyle \pi_2^0} \\ \mathbb{Z}_2 & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z}_2 \end{array} \qquad \begin{array}{ccc} \mathbb{Z}_{\gcd(2,n)} & \xrightarrow{\alpha_n^1} & \mathbb{Z}_{\gcd(2,n)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}_n \oplus \mathbb{Z}_2 & \xrightarrow{\alpha_n^0} & \mathbb{Z}_n \oplus \mathbb{Z}_2 \\ \left(\frac{m}{n}, \mathrm{id}_{\mathbb{Z}_2}\right)\big\downarrow\big\uparrow\left(\pi_{n,m}, \frac{m}{n}\right) & & \left(\frac{m}{n}, \mathrm{id}_{\mathbb{Z}_2}\right)\big\downarrow\big\uparrow\left(\pi_{n,m}, \frac{m}{n}\right) \\ \mathbb{Z}_m \oplus \mathbb{Z}_2 & \xrightarrow[\alpha_m^0]{} & \mathbb{Z}_m \oplus \mathbb{Z}_2 \end{array} \qquad \begin{array}{ccc} \mathbb{Z}_{\gcd(2,n)} & \xrightarrow{\alpha_n^1} & \mathbb{Z}_{\gcd(2,n)} \\ \times\frac{\gcd(2,m)}{\gcd(2,n)}\big\downarrow\big\uparrow\pi_{\gcd(2,n),\gcd(2,m)} & & \times\frac{\gcd(2,m)}{\gcd(2,n)}\big\downarrow\big\uparrow\pi_{\gcd(2,n),\gcd(2,m)} \\ \mathbb{Z}_{\gcd(2,m)} & \xrightarrow[\alpha_m^1]{} & \mathbb{Z}_{\gcd(2,m)} \end{array}$$

Opposing our previous traditions we start with the right diagrams first, as they are almost trivial. Note the upper right diagram forces α_n^1 to be the identity for all $n \geq 2$, which also makes the central right and bottom right diagrams commute. For the left diagram we will consider α_n^0 as a matrix

$$\alpha_n^0 = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

where a_n, b_n, c_n, d_n are homomorphisms. For the upper left diagram to commute we must have

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} [x]_n \\ 0 \end{pmatrix} = \begin{pmatrix} [x]_n \\ 0 \end{pmatrix},$$

which forces $a_n = 1$ and $c_n = 0$. For the central diagram to commute

$$\pi_0 \left(\begin{pmatrix} 1 & b_n \\ 0 & d_n \end{pmatrix} \begin{pmatrix} [x]_n \\ [y]_2 \end{pmatrix} \right) = [y]_2,$$

which forces $d_n = 1$. So the only homomorphism left is $b_n : \mathbb{Z}_2 \to \mathbb{Z}_n$ hence the bottom left diagram commutes whenever

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{b_n} & \mathbb{Z}_n \\ \mathrm{id}_{\mathbb{Z}_2}\big\downarrow\big\uparrow\times\frac{m}{n} & & \times\frac{m}{n}\big\downarrow\big\uparrow\pi_{n,m} \\ \mathbb{Z}_2 & \xrightarrow[b_m]{} & \mathbb{Z}_m \end{array}$$

commutes. There are two possible homomorphisms between $\mathbb{Z}_2$ and $\mathbb{Z}_n$, which is 0 and $\frac{n}{2}$. Trivially 0 makes the diagram commute, we check if $\times\frac{n}{2}$ makes the diagram commute.

$$\begin{aligned} \times\frac{n}{2}\left(\times\frac{m}{n}([x]_2)\right) &= \left[\frac{m}{2}x\right]_n = \pi_{n,m}\left(\times\frac{m}{2}([x]_2)\right), \\ \times\frac{m}{n}\left(\times\frac{n}{2}([x]_2)\right) &= \left[\frac{m}{2}x\right]_m = \times\frac{m}{2}([x]_2). \end{aligned}$$

So α_n^0 can either be the identity or $\begin{pmatrix} 1 & \times\frac{n}{2} \\ 0 & 1 \end{pmatrix}$. We find our possible β maps to which we have the diagram

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\iota_1} & \mathbb{T} \oplus \mathbb{Z}_2 & \xrightarrow{\pi_2} & \mathbb{Z}_2 \\ \mathrm{id}_{\mathbb{R}}\big\downarrow & & \big\downarrow\beta & & \big\downarrow\mathrm{id}_{\mathbb{Z}_2} \\ \mathbb{R} & \xrightarrow[\iota_1]{} & \mathbb{T} \oplus \mathbb{Z}_2 & \xrightarrow[\pi_2]{} & \mathbb{Z}_2. \end{array}$$

Similarly to computing the Λ-morphisms we will write β as a matrix of group homomorphisms

$$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and check what conditions must hold such that the diagram commutes. If the left square commutes then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} [x]_{\mathbb{R}/\mathbb{Z}} \\ 0 \end{pmatrix} = \begin{pmatrix} [x]_{\mathbb{R}/\mathbb{Z}} \\ 0 \end{pmatrix},$$

which forces $a = 1$ and $c = 0$. If the right square commutes we must have

$$\pi_2\left(\begin{pmatrix}1 & b\\ 0 & d\end{pmatrix}\begin{pmatrix}[x]_{\mathbb{R}/\mathbb{Z}}\\ y\end{pmatrix}\right) = y,$$

forcing $d = 1$. Consider the last diagram

$$\begin{array}{ccc}
\mathbb{Z}_n \oplus \mathbb{Z}_2 & \xrightarrow{(\frac{1}{n}\overline{\rho},\mathrm{id}_{\mathbb{Z}_2})} & \mathbb{T} \oplus \mathbb{Z}_2 \\
{\scriptstyle \alpha_n^0}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
\mathbb{Z}_n \oplus \mathbb{Z}_2 & \xrightarrow[(\frac{1}{n}\overline{\rho},\mathrm{id}_{\mathbb{Z}_2})]{} & \mathbb{T} \oplus \mathbb{Z}_2.
\end{array}$$

As the only wriggle room we have is in the top right corner both for α_n^0 and β, the diagram above will commute whenever

$$\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{\mathrm{id}_{\mathbb{Z}_2}} & \mathbb{Z}_2 \\
{\scriptstyle b_n}\big\downarrow & & \big\downarrow{\scriptstyle b} \\
\mathbb{Z}_n & \xrightarrow[\frac{1}{n}\overline{\rho}]{} & \mathbb{T}
\end{array}$$

commutes. As the top homomorphism is the identity, if b makes the diagram commute then $b = \frac{1}{n}\overline{\rho} \circ b_n$. As we already found the possible b_n's, b is either 0 or $\frac{1}{2}\overline{\rho}$, hence β is either the identity matrix or $\begin{pmatrix}1 & \frac{1}{2}\overline{\rho}\\ 0 & 1\end{pmatrix}$ which finishes the case of both n and m being even. Now for the case of n, m both being odd. The six diagrams from the previous case becomes

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\mathrm{id}_{\mathbb{Z}}} & \mathbb{Z} \\
{\scriptstyle \pi_n}\big\downarrow & & \big\downarrow{\scriptstyle \pi_n} \\
\mathbb{Z}_n & \xrightarrow[\alpha_n^0]{} & \mathbb{Z}_n
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{\mathrm{id}_{\mathbb{Z}_2}} & \mathbb{Z}_2 \\
\big\downarrow & & \big\downarrow \\
0 & \xrightarrow[\alpha_n^1]{} & 0
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Z}_n & \xrightarrow{\alpha_n^0} & \mathbb{Z}_n \\
{\scriptstyle \pi_{2,n}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{2,n}} \\
\mathbb{Z}_2 & \xrightarrow[\mathrm{id}_{\mathbb{Z}_2}]{} & \mathbb{Z}_2
\end{array}
\qquad
\begin{array}{ccc}
0 & \xrightarrow{\alpha_n^1} & 0 \\
\big\downarrow & & \big\downarrow \\
0 & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Z}_n & \xrightarrow{\alpha_n^0} & \mathbb{Z}_n \\
{\scriptstyle \times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle \pi_{n,m}} & & {\scriptstyle \times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle \pi_{n,m}} \\
\mathbb{Z}_m & \xrightarrow[\alpha_m^0]{} & \mathbb{Z}_m
\end{array}
\qquad
\begin{array}{ccc}
0 & \xrightarrow{\alpha_n^1} & 0 \\
\big\downarrow\big\uparrow & & \big\downarrow\big\uparrow \\
0 & \xrightarrow[\alpha_m^1]{} & 0
\end{array}$$

In this case α_n^0 has to be the identity as that is the only group isomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_n$, and α_n^1 is the identity on 0. For the β map, the diagram

$$\begin{array}{ccccc}
\mathbb{R} & \xrightarrow{\iota_1} & \mathbb{T} \oplus \mathbb{Z}_2 & \xrightarrow{\pi_2} & \mathbb{Z}_2 \\
{\scriptstyle \mathrm{id}_{\mathbb{R}}}\big\downarrow & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}_2}} \\
\mathbb{R} & \xrightarrow[\iota_1]{} & \mathbb{T} \oplus \mathbb{Z}_2 & \xrightarrow[\pi_2]{} & \mathbb{Z}_2
\end{array}$$

does not depend on n or m hence $\beta = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and as before we need to find the b's making

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ {\scriptstyle b_n}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ \mathbb{Z}_n & \xrightarrow[\frac{1}{n}\overline{\rho}]{} & \mathbb{T} \end{array}$$

commute. One sees this commutes for all group homomorphisms $b : \mathbb{Z}_2 \to \mathbb{T}$, for which there are two, namely 0 and $\times\frac{1}{2}\overline{\rho}$, so β is again either the identity or $\begin{pmatrix} 1 & \frac{1}{2}\overline{\rho} \\ 0 & 1 \end{pmatrix}$. We now consider the mixed case of n being odd and m being even. The only thing that changes in the Λ-morphism case is bottom two diagrams:

$$\begin{array}{ccc} \mathbb{Z}_n \oplus 0 & \xrightarrow{\alpha_n^0} & \mathbb{Z}_n \oplus 0 \\ {\scriptstyle (\times\frac{m}{n},0)}\big\downarrow\big\uparrow{\scriptstyle (\pi_{n,m},0)} & & {\scriptstyle (\times\frac{m}{n},0)}\big\downarrow\big\uparrow{\scriptstyle (\pi_{n,m},0)} \\ \mathbb{Z}_m \oplus \mathbb{Z}_2 & \xrightarrow[\alpha_m^0]{} & \mathbb{Z}_m \oplus \mathbb{Z}_2, \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\alpha_n^1} & 0 \\ \big\downarrow\big\uparrow & & \big\downarrow\big\uparrow \\ \mathbb{Z}_2 & \xrightarrow[\alpha_m^1]{} & \mathbb{Z}_2. \end{array}$$

Note α_n^1 has to be the identity on $\mathbb{Z}_2$ and again α_n^0 has to be a strictly upper triangular matrix, hence the left diagram commutes whenever

$$\begin{array}{ccc} 0 & \xrightarrow{b_n} & \mathbb{Z}_n \\ \big\downarrow\big\uparrow & & {\scriptstyle \times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle \pi_{n,m}} \\ \mathbb{Z}_2 & \xrightarrow[b_m]{} & \mathbb{Z}_m \end{array}$$

commutes. This trivially commutes when $b_m = 0$, we claim it also commutes for $b_m = \times\frac{m}{2}$. Indeed as n divides m then $m = kn$ for some $k \in \mathbb{Z}$, but as m is even and n is odd, we must have that k is even as well. Hence

$$\times\frac{m}{2} = \frac{k}{2}n,$$

so the image of $\times\frac{m}{2}$ is contained in the kernel of $\pi_{n,m}$, and so the diagram commutes. So α_n^0 is either the identity or $\begin{pmatrix} 1 & \times\frac{n}{2} \\ 0 & 1 \end{pmatrix}$. For the β, we only deal with either n or m never a mix, hence we will be in one of the previous cases. Thus we have found all possible automorphisms of $\underline{\mathrm{K}}\mathrm{T}_\mathrm{u}$, namely

$$\mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T}_\mathrm{u}(A)) = \left\{ \begin{array}{c} (\mathrm{id}_\mathbb{Z}, \mathrm{id}_{\mathbb{Z}_2}, (\mathrm{id}_{\mathbb{Z}_n \oplus \mathbb{Z}_n[n]}, \mathrm{id}_{\mathbb{Z}_{\gcd 2,n}})_{n \geq 2}, \mathrm{id}_X, \mathrm{id}_{\mathbb{T} \oplus \mathbb{Z}_2}), \\ (\mathrm{id}_\mathbb{Z}, \mathrm{id}_{\mathbb{Z}_2} \left(\begin{pmatrix} 1 & \times\frac{n}{2} \\ 0 & 1 \end{pmatrix}, \mathrm{id}_{Z_{\gcd(2,n)}} \right)_{n \geq 2}, \mathrm{id}_X, \begin{pmatrix} 1 & \frac{1}{2}\overline{\rho} \\ 0 & 1 \end{pmatrix} \end{array} \right\} \cong \mathbb{Z}_2.$$

By Theorem 8.2 $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A) \cong \mathbb{Z}_2$.

The astute reader probably noticed, in the example we did not need to consider all diagrams when calculating the automorphism group. We needed only to consider the diagrams that correspond to the upper triangular part of the matrices. This is no coincidence, and the last part of the thesis will be devoted to examining this.

Definition 8.5. Let $K = (G_0, g, G_1, X, \xi)$ be a $\mathcal{KT}_u$-system, and let $N(K)$ be the abelian group of families of group homomorphisms

$$(\varphi, (\psi_n^i)_{i\in\{0,1\},n\geq 2}) \in \mathrm{Hom}(G_1, X/\overline{\xi(G_0)}) \times \prod_{\substack{i\in\{0,1\}\\ n\geq 2}} \mathrm{Hom}(G_{1-i}[n], G_i/nG_1)$$

such that

$$\begin{array}{ccc} G_{1-i}[n] & \xrightarrow{\psi_n^i} & G_i/nG_i \\ {\scriptstyle\iota_{m,n}^{1-i}}\big\downarrow\big\uparrow{\scriptstyle\times\frac{m}{n}} & & {\scriptstyle\times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle\pi_{n,m}^i} \\ G_{1-i}[m] & \xrightarrow{\psi_m^i} & G_i/mG_i \end{array}$$

commutes for $i \in \{0,1\}$, $n, m \geq 2$ such that $n|m$ further

$$\begin{array}{ccc} G_1[n] & \hookrightarrow & G_1 \\ {\scriptstyle\psi_n^i}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ G_0/nG_0 & \xrightarrow[\frac{1}{n}\overline{\xi}]{} & X/\overline{\xi(G_0)} \end{array}$$

commutes for all $n \geq 2$.

Lets look at the relationship between KT_u-automorphisms and automorphisms on $N(K)$.

Proposition 8.6. *Let $K = (G_0, g, G_1, X, \xi)$ be a $\mathcal{KT}_u$-system, and $(\alpha^0, \alpha^1, \gamma) \in \mathrm{Aut}(K)$ be a $\mathcal{KT}_u$-automorphism. Then $\Psi_{\alpha^i,\gamma} : N(K) \to N(K)$ given by*

$$\Psi_{\alpha^i,\gamma}(\varphi, (\psi_n^i)_{i\in\{0,1\},n\geq 2}) = \left(\overline{\gamma} \circ \varphi \circ \left(\alpha^1\right)^{-1}, \left(\overline{\alpha}_n^i \circ \psi_n^i \circ \left(\widetilde{\alpha}_n^{1-i}\right)^{-1}\right)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right),$$

is a group isomorphism, where $\overline{\cdot}, \widetilde{\cdot}$ *denotes the induced quotient and n-torsion maps respectively.*

Proof. We show $\Psi_{\alpha^i,\gamma}$ is well defined. Note $\left(\overline{\gamma} \circ \varphi \circ \left(\alpha^1\right)^{-1}, \left(\overline{\alpha}_n^i \circ \psi_n^i \circ \left(\widetilde{\alpha}_n^{1-i}\right)^{-1}\right)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right)$ is an element of $\mathrm{Hom}(G_1, X/\overline{\xi(G_0)}) \times \prod_{\substack{i\in\{0,1\}\\ n\geq 2}} \mathrm{Hom}(G_{1-i}[n], G_i/nG_1)$. Consider the diagram

$$\begin{array}{ccc} G_{1-i}[n] & \xrightarrow{\overline{\alpha}_n^i\circ\psi_n^i\circ(\widetilde{\alpha}_n^{1-i})^{-1}} & G_i/nG_i \\ {\scriptstyle\iota_{m,n}^{1-i}}\big\downarrow\big\uparrow{\scriptstyle\times\frac{m}{n}} & & {\scriptstyle\times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle\pi_{n,m}^i} \\ G_{1-i}[m] & \xrightarrow{\overline{\alpha}_m^i\circ\psi_m^i\circ(\widetilde{\alpha}_m^{1-i})^{-1}} & G_i/mG_i \end{array}$$

and expand it to

$$\begin{array}{ccccccc} G_{1-i}[n] & \xrightarrow{(\widetilde{\alpha}_n^{1-i})^{-1}} & G_{1-i}[n] & \xrightarrow{\psi_n^i} & G_i/nG_i & \xrightarrow{\overline{\alpha}_n^i} & G_i/nG_i \\ {\scriptstyle\iota_{m,n}^{1-i}}\big\downarrow\big\uparrow{\scriptstyle\times\frac{m}{n}} & & {\scriptstyle\iota_{m,n}^{1-i}}\big\downarrow\big\uparrow{\scriptstyle\times\frac{m}{n}} & & {\scriptstyle\times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle\pi_{n,m}^i} & & {\scriptstyle\times\frac{m}{n}}\big\downarrow\big\uparrow{\scriptstyle\pi_{n,m}^i} \\ G_{1-i}[m] & \xrightarrow{(\widetilde{\alpha}_m^{1-i})^{-1}} & G_{1-i}[m] & \xrightarrow{\psi_m^i} & G_i/mG_i & \xrightarrow{\overline{\alpha}_m^i} & G_i/mG_i \end{array}$$

Breaking tradition we observe the central square commutes by definition of $N(K)$, now we consider the left square. It commutes with $\iota_{m,n}^{1-i}$ by definition of $(\widetilde{\alpha}_n^{1-i})^{1-i}$, we show the other direction. Note

$$\begin{array}{ccc} G_{1-i}[m] & \xrightarrow{\times \frac{m}{n}} & G_{1-i}[n] \\ {\scriptstyle \iota_m}\big\downarrow & & \big\downarrow{\scriptstyle \iota_n} \\ G_{1-i} & \xrightarrow[\times \frac{m}{n}]{} & G_{1-i}. \end{array}$$

commutes, thus

$$\iota_n \circ \times \frac{m}{n} \circ (\widetilde{\alpha}_m^{1-i})^{-1} = \times \frac{m}{n} \circ \iota_m \circ (\widetilde{\alpha}_m^{1-i})^{-1} = \times \frac{m}{n} \circ (\alpha^1)^{-1} \circ \iota_m.$$

Since α^{1-i} is a group homomorphism $\times \frac{m}{n} \circ (\alpha^{1-i})^{-1} = (\alpha^{1-i})^{-1} \circ \times \frac{m}{n}$, continuing

$$(\alpha^{1-i})^{-1} \circ \times \frac{m}{n} \circ \iota_m = (\alpha^{1-i})^{-1} \circ \circ \iota_n \circ \times \frac{m}{n} = \iota_n \circ (\widetilde{\alpha}_n^{1-i})^{-1} \circ \times \frac{m}{n}.$$

By injectivity of ι_n the square commutes. The right square commutes with respect to $\pi_{n,m}$ by definition of $\overline{\alpha}_n^i$, for the other direction note

$$\begin{array}{ccc} G_i & \xrightarrow{\times \frac{m}{n}} & G_i \\ {\scriptstyle \pi_m}\big\downarrow & & \big\downarrow{\scriptstyle \pi_n} \\ G_i/mG_i & \xrightarrow[\times \frac{m}{n}]{} & G_i/nG_i \end{array}$$

commutes. Hence

$$\times \frac{m}{n} \circ \overline{\alpha}_m^i \circ \pi_m = \times \frac{m}{n} \circ \pi_m \circ \alpha^i = \pi_n \circ \times \frac{m}{n} \circ \alpha^i,$$

as before $\times \frac{m}{n} \circ \alpha^i = \alpha^i \circ \times \frac{m}{n}$ continuing our calculation

$$\pi_n \circ \alpha^i \circ \times \frac{m}{n} = \overline{\alpha}_n^i \circ \pi_n \circ \times \frac{m}{n} = \overline{\alpha}_n^i \circ \times \frac{m}{n} \circ \pi_m.$$

By surjectivity of π_m the right square commutes. We wish to expand the diagram

$$\begin{array}{ccc} G_1[n] & \hookrightarrow & G_1 \\ {\scriptstyle \overline{\alpha}_n^0 \circ \psi_n^0 (\alpha_n^{1-i})^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\gamma} \circ \varphi \circ (\widetilde{\alpha}^1)^{-1}} \\ G_0/nG_0 & \xrightarrow[\times \frac{m}{n}]{} & X/\overline{\xi(G_0)}, \end{array}$$

but to ease readability of the expanded diagram, we first reflect it over the main diagonal

$$\begin{array}{ccccccc} G_1[n] & \xrightarrow{(\widetilde{\alpha}_n^1)^{-1}} & G_1[n] & \xrightarrow{\psi_n^0} & G_0/nG_0 & \xrightarrow{\overline{\alpha}_n^0} & G_0/nG_0 \\ \big\downarrow & & \big\downarrow & & {\scriptstyle \frac{1}{n}\overline{\xi}}\big\downarrow & & \big\downarrow{\scriptstyle \frac{1}{n}\overline{\xi}} \\ G_1 & \xrightarrow[(\alpha^1)^{-1}]{} & G_1 & \xrightarrow[\varphi]{} & X/\overline{\xi(G_0)} & \xrightarrow[\overline{\gamma}]{} & X/\overline{\xi(G_0)}. \end{array}$$

Again the central square commutes by definition of $N(K)$. The left square commutes by definition of $(\widetilde{\alpha}_n^1)^{-1}$, and the right square commutes since $(\alpha^0, \alpha^1, \gamma)$ is a KT_u-automorphism. Hence $\Psi_{\alpha^1,\gamma}$

is well defined. We show $\Psi_{\alpha^i,\gamma}$ is a bijection.
Surjectivity: Let $(\varphi, (\psi_n^i)_{i\in\{0,1\},n\geq 2})) \in N(K)$, and consider

$$\left(\overline{\gamma}^{-1} \circ \varphi \circ \alpha^1, \left(\left(\overline{\alpha}_n^i\right)^{-1} \circ \psi_n^i \circ \widetilde{\alpha}_n^{1-i}\right)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right),$$

which is an element of $N(K)$, to see this apply the arguments above with notational changes. Then

$$\begin{aligned}
&\Psi_{\alpha^i,\gamma}\left(\overline{\gamma}^{-1} \circ \varphi \circ \alpha^1, \left(\left(\overline{\alpha}_n^i\right)^{-1} \circ \psi_n^i \circ \widetilde{\alpha}_n^{1-i}\right)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right)\\
&= \left(\overline{\gamma} \circ \overline{\gamma}^{-1} \circ \varphi \circ \alpha^1 \circ (\alpha^1)^{-1}, \left(\overline{\alpha}_n^i \circ \left(\overline{\alpha}_n^i\right)^{-1} \circ \psi_n^i \circ \widetilde{\alpha}_n^{1-i} \circ \left(\widetilde{\alpha}_n^{1-i}\right)^{-1}\right)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right)\\
&= (\varphi, (\psi_n^i)_{i\in\{0,1\},n\geq 2})).
\end{aligned}$$

Injectivity: Let $(\varphi_1, (\psi_{1,n}^i)_{i\in\{0,1\},n\geq 2}), (\varphi_2, (\psi_{2,n}^i)_{i\in\{0,1\},n\geq 2}) \in N(K)$ such that for $i \in \{0,1\}$ and $n \geq 2$

$$\overline{\gamma} \circ \varphi_1 \circ (\alpha^1)^{-1} = \overline{\gamma} \circ \varphi_2 \circ (\alpha^1)^{-1}$$

and,

$$\widetilde{\alpha}_n^i \circ \psi_{1,n}^i \circ (\overline{\alpha}_n^{1-i})^{-1} = \widetilde{\alpha}_n^i \circ \psi_{2,n}^i \circ (\overline{\alpha}_n^{1-i})^{-1}.$$

Since α^0, α^1 and γ are isomorphisms we compose with their inverses on both sides. Thus $\varphi_1 = \varphi_2$, and $\psi_{1,n} = \psi_{2,n}$. We show $\Psi_{\alpha^i,\gamma}$ is a group homomorphism.
Let $(\varphi_1, (\psi_{1,n}^i)_{i\in\{0,1\},n\geq 2}), (\varphi_2, (\psi_{2,n}^i)_{i\in\{0,1\},n\geq 2}) \in N(K)$, for $i \in \{0,1\}$ and $n \geq 2$

$$\begin{aligned}
&\Psi_{\alpha^i,\gamma}((\varphi_1, (\psi_{1,n}^i)) + (\varphi_2, \psi_{2,n}^i))\\
&= \Psi_{\alpha^i,\gamma}(\varphi_1 + \varphi_2, (\psi_{1,n}^i + \psi_{2,n}^i))\\
&= (\overline{\gamma} \circ (\varphi_1 + \varphi_2) \circ (\alpha^1)^{-1}, (\overline{\alpha}_n^i \circ (\psi_{1,n}^i + \psi_{2,n}^i) \circ (\widetilde{\alpha}_n^{1-i})^{-1}))\\
&= (\overline{\gamma} \circ \varphi_1 \circ (\alpha^1)^{-1} + \overline{\gamma} \circ \varphi_2 \circ (\alpha^1)^{-1}, (\overline{\alpha}_n^i \circ \psi_{1,n}^i \circ (\widetilde{\alpha}_n^{1-i})^{-1} + \overline{\alpha}_n^i \circ \psi_{2,n}^i \circ (\widetilde{\alpha}_n^{1-i})^{-1}))\\
&= (\overline{\gamma} \circ \varphi_1 \circ (\alpha^1)^{-1}, (\overline{\alpha}_n^i \circ \psi_{1,n}^i \circ (\widetilde{\alpha}_n^{1-i})^{-1})) + (\overline{\gamma} \circ \varphi_2 \circ (\alpha^1)^{-1}, (\overline{\alpha}_n^i \circ \psi_{2,n}^i) \circ (\widetilde{\alpha}_n^{1-i})^{-1}))\\
&= \Psi_{\alpha^i,\gamma}(\varphi_1, (\psi_{1,n}^i)) + \Psi_{\alpha^i,\gamma}(\varphi_2, (\psi_{2,n}^i)).
\end{aligned}$$

so $\Psi_{\alpha^i,\gamma}$ an automorphism on $N(K)$. □

Our final goal is to describe the automorphisms on $\underline{\mathcal{KT}}_u$-systems using only $\mathcal{KT}_u$. To do this we first need to know what a semidirect product is.

Definition 8.7 ([DF04], Chap 5, Theorem 10)**.** Let G and H be groups and let $\varphi : G \to \mathrm{Aut}(H)$ be a group homomorphism. Construct a group which set is $H \times G$ and with binary operation given by

$$(h_1, g_1)(h_2, g_2) = (h_1\varphi_{g_1}(h_2), g_1 g_2).$$

We call this group the *semidirect product*, and denote it by $H \rtimes_\varphi G$.

After a definition it is always nice to see an example.

Example 8.8. Let $K = (G_0, g, G_1, X, \xi)$ be a $\mathcal{KT}_u$-system then $N(K) \rtimes_\Delta \mathrm{Aut}(K)$ with $\Delta : \mathrm{Aut}(K) \to \mathrm{Aut}(N(K))$ given by $\Delta(\alpha^0, \alpha^1, \gamma) = \Psi_{\alpha^i,\gamma}$ forms a semidirect product.

Proof. Δ is well defined by Proposition 8.6, we show Δ is a group homomorphism.
Let $(\alpha^0, \alpha^1, \gamma_1), (\beta^0, \beta^1, \gamma_2) \in \mathrm{Aut}(K)$.

$$\begin{aligned}
&\Delta((\alpha^0, \alpha^1, \gamma_1) \circ (\beta^0, \beta^1, \gamma_2)) \left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= \Delta(\alpha^0 \circ \beta^0, \alpha^1 \circ \beta^1, \gamma_1 \circ \gamma_2) \left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= \Psi_{\alpha^i\circ\beta^i, \gamma_1\circ\gamma_2} \left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= \left(\overline{\gamma}_1 \circ \overline{\gamma}_2 \circ \varphi \circ (\beta^1)^{-1} \circ (\alpha^1)^{-1}, \overline{\alpha}_n^i \circ \overline{\beta}_n^i \circ \psi_n^i \circ (\widetilde{\beta}_n^{1-i})^{-1} \circ (\widetilde{\alpha}_n^{1-i})^{-1})_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= \Psi_{\alpha^i, \gamma_1} \left(\overline{\gamma}_2 \circ \varphi \circ (\beta^1)^{-1}, (\overline{\beta}_n^i \circ \psi_n^i \circ (\widetilde{\beta}_n^{1-i})^{-1})_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= (\Psi_{\alpha^i,\gamma_1} \circ \Psi_{\beta^i,\gamma_2}) \left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\}\\ n\geq 2}}\right) \\
&= \Delta(\alpha^0, \alpha^1, \gamma) \circ \Delta(\beta^0, \beta^1, \xi) \left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\}\\ n\geq 2}}\right).
\end{aligned}$$

By Definition 8.7 we can form the semidirect product $N(K) \rtimes_\Delta \mathrm{Aut}(K)$. □

The next bit of the thesis is about proving the automorphism group of $T(K)$, is given by the semidirect product from Example 8.8. Before we can see that proof, we will need one more fact about semidirect products of groups.

Proposition 8.9. *Let*

$$1 \longrightarrow G \xrightarrow{\varphi} H \underset{s}{\overset{\psi}{\rightleftarrows}} L \longrightarrow 1$$

be a short split exact sequence of groups. Then $H \cong G \rtimes L$, with action induced from the split.

We are now ready to prove our last theorem.

Theorem 8.10. *Let $K = (G_0, g, G_1, X, \xi)$ be a $\mathcal{KT}_u$-system and $\mathbb{K} = T(K)$ be it's induced trivial system. Then there exists a group homomorphism θ such that the sequence*

$$1 \longrightarrow N(K) \xrightarrow{\theta} \mathrm{Aut}(\mathbb{K}) \xrightarrow{\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u}} \mathrm{Aut}(K) \longrightarrow 1,$$

is short exact, and admits a splitting whose induced action $\mathrm{Aut}(K) \curvearrowright N(K)$ is Δ from Example 8.8. In particular $\mathrm{Aut}(\mathbb{K}) \cong N(K) \rtimes_\Delta \mathrm{Aut}(K)$.

Proof. Consider the map $\theta : N(K) \to \mathrm{Aut}(\mathbb{K})$ given by

$$\theta\left(\varphi, (\psi_{i\in\{0,1\} n\geq 2})\right) = \left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, (\alpha_n^i)_{i\in\{0,1\}, n\geq 2}\right), \mathrm{id}_X, \beta\right),$$

where $\alpha_n^i = \begin{pmatrix} \mathrm{id}_{G_i/nG_i} & \psi_n^i \\ 0 & \mathrm{id}_{G_{1-i}[n]} \end{pmatrix}_{i\in\{0,1\}, n\geq 2}$ and $\beta = \begin{pmatrix} \mathrm{id}_{X/\overline{\xi(G_0)}} & \varphi \\ 0 & \mathrm{id}_{G_1} \end{pmatrix}$. Both α_n^i and β are invertible, hence to show θ is well defined it suffices to show $\mathrm{Im}(\theta) \subseteq \mathrm{Hom}(\mathbb{K})$. We show

$\big(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, (\alpha_n^i)_{i\in\{0,1\},n\geq 2}\big)$ is a Λ-morphism. Both

$$\begin{array}{ccc} G_i & \xrightarrow{\mathrm{id}_{G_i}} & G_i \\ {\scriptstyle \mu_n^i}\downarrow & & \downarrow{\scriptstyle \mu_n^i} \\ G_i/nG_i \oplus G_{1-i}[n] & \xrightarrow[\alpha_n^i]{} & G_i/nG_i \oplus G_{1-i}[n] \end{array} \quad , \quad \begin{array}{ccc} G_i/nG_i \oplus G_{1-i}[n] & \xrightarrow{\alpha_n^i} & G_i/nG_i \oplus G_{1-i}[n] \\ {\scriptstyle \nu_n^i}\downarrow & & \downarrow{\scriptstyle \nu_n^i} \\ G_{1-i} & \xrightarrow[\mathrm{id}_{G_{1-i}}]{} & G_{1-i}. \end{array}$$

commutes as we have the identities on the diagonal of α_n^i. Further as 0 in the bottom left corner the diagram

$$\begin{array}{ccc} G_i/nG_i \oplus G_{1-i}[n] & \xrightarrow{\alpha_n^i} & G_i/nG_i \oplus G_{1-i}[n] \\ {\scriptstyle (\times\frac{m}{n},\iota_{m,n}^{1-i})}\downarrow\uparrow{\scriptstyle (\pi_{n,m}^i,\times\frac{m}{n})} & & {\scriptstyle (\times\frac{m}{n},\iota_{m,n}^{1-i})}\downarrow\uparrow{\scriptstyle (\pi_{n,m}^i,\times\frac{m}{n})} \\ G_i/mG_i \oplus G_{1-i}[m] & \xrightarrow[\alpha_m^i]{} & G_i/mG_i \oplus G_{1-i}[m]. \end{array}$$

commutes if and only if

$$\begin{array}{ccc} G_{1-i}[n] & \xrightarrow{\psi_n^i} & G_i/nG_i \\ {\scriptstyle \iota_{m,n}^{1-i}}\downarrow\uparrow{\scriptstyle \times\frac{m}{n}} & & {\scriptstyle \times\frac{m}{n}}\downarrow\uparrow{\scriptstyle \pi_{n,m}^i} \\ G_{1-i}[m] & \xrightarrow{\psi_m^i} & G_i/mG_i \end{array}$$

commutes. Indeed this does commute as $\big(\varphi, \big(\psi_{i\in\{0,1\}n\geq 2}\big)\big) \in N(K)$.
We show β makes $\big(\big(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, (\alpha_n^i)_{i\in\{0,1\},n\geq 2}\big), \mathrm{id}_X, \beta\big)$ into a $\underline{\mathcal{K}}\mathcal{T}_u$-morphism. Consider

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\xi} & X & \xrightarrow{\upsilon} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow{\psi} & G_1 \\ {\scriptstyle \mathrm{id}_{G_0}}\downarrow & & {\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_1}} \\ G_0 & \xrightarrow[\xi]{} & X & \xrightarrow[\upsilon]{} & X/\overline{\xi(G_0)} \oplus G_1 & \xrightarrow[\psi]{} & G_1 \end{array}$$

which commutes as we have the identities on the diagonal of β. Again as 0 is in the bottom left corner the diagram

$$\begin{array}{ccc} G_0/nG_0 \oplus G_1[n] & \xrightarrow{(\frac{1}{n}\overline{\xi},\iota_n^1)} & X/\overline{\xi(G_0)} \oplus G_1 \\ {\scriptstyle \alpha_n^0}\downarrow & & {\scriptstyle \beta}\downarrow \\ G_0/nG_0 \oplus G_1[n] & \xrightarrow{(\frac{1}{n}\overline{\xi},\iota_n^1)} & X/\overline{\xi(G_0)} \oplus G_1 \end{array}$$

commutes if and only if

$$\begin{array}{ccc} G_1[n] & \hookrightarrow & G_1 \\ {\scriptstyle \psi_n^i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G_0/nG_0 & \xrightarrow[\frac{1}{n}\overline{\xi}]{} & X/\overline{\xi(G_0)} \end{array}$$

commutes. Which again is true as $\big(\varphi, \big(\psi_{i\in\{0,1\}n\geq 2}\big)\big) \in N(K)$. We show θ is a group homomorphism. Let $\left(\varphi_1, (\psi^i_{1,n})_{i\in\{0,1\},n\geq 2}\right), \left(\varphi_2, (\psi^i_{2,n})_{i\in\{0,1\},n\geq 2}\right) \in N(K)$.

$$\begin{aligned}
&\theta\left(\varphi_1, (\psi^i_{1,n})_{i\in\{0,1\},n\geq 2}\right)\circ\theta\left(\varphi_2, (\psi^i_{2,n})_{i\in\{0,1\},n\geq 2}\right)\\
&=\left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \left(\alpha^i_{1,n}\circ\alpha^i_{2,n}\right)_{i\in\{0,1\},n\geq 2}\right), \mathrm{id}_X, \beta_1\circ\beta_2\right)\\
&=\left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \left(\begin{pmatrix}\mathrm{id}_{G_i/nG_i} & \psi^i_{2,n}+\psi^i_{1,n}\\ 0 & \mathrm{id}_{G_{1-i}[n]}\end{pmatrix}\right)_{i\in\{0,1\},n\geq 2}\right), \mathrm{id}_X, \begin{pmatrix}\mathrm{id}_{X/\overline{\xi(G_0)}} & \varphi_2+\varphi_1\\ 0 & \mathrm{id}_{G_1}\end{pmatrix}\right)\\
&\theta\left(\varphi_1+\varphi_2, (\psi^i_{1,n}+\psi_{2,n})_{i\in\{0,1\},n\geq 2}\right).
\end{aligned}$$

We show

$$1 \longrightarrow N(K) \xrightarrow{\theta} \mathrm{Aut}(\mathbb{K}) \xrightarrow{\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u}} \mathrm{Aut}(K) \longrightarrow 1,$$

is exact. By Theorem 7.22 $\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u}$ is surjective, and θ is injective as $\ker(\theta)=0$. Note

$$\ker(\widetilde{F}_{\underline{\mathcal{K}}\mathcal{T}_u}) = ((\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \alpha^i_n)_{i\in\{0,1\},n\geq 2}, \mathrm{id}_X, \beta),$$

for any α^i_n making $(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \alpha^i_n)_{i\in\{0,1\},n\geq 2}$ into a Λ-isomorphism and β making $((\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \alpha^i_n)_{i\in\{0,1\},n\geq 2}, \mathrm{id}_X, \beta)$ into a $\underline{\mathcal{K}}\mathcal{T}_u$-isomorphism.
As $((\mathrm{id}_{G_0}, \mathrm{id}_{G_1}\alpha^i_n)_{i\in\{0,1\},n\geq 2}, \mathrm{id}_X, \beta)$ is a $\underline{\mathcal{K}}\mathcal{T}_u$-isomorphism the diagrams

$$\begin{array}{ccc}
G_i & \xrightarrow{\mathrm{id}_{G_i}} & G_i\\
{\scriptstyle \mu^i_n}\downarrow & & \downarrow{\scriptstyle \mu^i_n}\\
G_i/nG_i\oplus G_{1-i}[n] & \xrightarrow[\alpha^i_n]{} & G_i/nG_i\oplus G_{1-i}[n],
\end{array}
\qquad
\begin{array}{ccc}
G_i/nG_i\oplus G_{1-i}[n] & \xrightarrow[\alpha^i_n]{} & G_i/nG_i\oplus G_{1-i}[n]\\
{\scriptstyle \mu^i_n}\downarrow & & \downarrow{\scriptstyle \mu^i_n}\\
G_{1-i} & \xrightarrow[\mathrm{id}_{G_{1-i}}]{} & G_{1-i},
\end{array}$$

commutes, hence $\alpha^i_n = \begin{pmatrix}\mathrm{id}_{G_i/nG_i} & b^i_n\\ 0 & \mathrm{id}_{G_{1-i}[n]}\end{pmatrix}$. Further the diagram

$$\begin{array}{ccc}
G_i/nG_i\oplus G_{1-i}[n] & \xrightarrow{\alpha^i_n} & G_i/nG_i\oplus G_{1-i}[n]\\
{\scriptstyle (\times\frac{m}{n},\iota^{1-i}_{m,n})}\downarrow\uparrow{\scriptstyle (\pi^i_{n,m},\times\frac{m}{n})} & & {\scriptstyle (\times\frac{m}{n},\iota^{1-i}_{m,n})}\downarrow\uparrow{\scriptstyle (\pi^i_{n,m},\times\frac{m}{n})}\\
G_i/mG_i\oplus G_{1-i}[m] & \xrightarrow[\alpha^i_m]{} & G_i/mG_i\oplus G_{1-i}[m].
\end{array}$$

commutes, hence b^i_n must satisfy

$$\begin{array}{ccc}
G_{1-i}[n] & \xrightarrow{b^i_n} & G_i/nG_i\\
{\scriptstyle \iota^{1-i}_{m,n}}\downarrow\uparrow{\scriptstyle \times\frac{m}{n}} & & {\scriptstyle \times\frac{m}{n}}\downarrow\uparrow{\scriptstyle \pi^i_{n,m}}\\
G_{1-i}[m] & \xrightarrow{b^i_m} & G_i/mG_i
\end{array}$$

whenever $n|m$. Also β makes the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\upsilon} & X/\overline{\xi(G_0)}\oplus G_1 & \xrightarrow{\psi} & G_1\\
{\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{G_1}}\\
X & \xrightarrow[\upsilon]{} & X/\overline{\xi(G_0)}\oplus G_1 & \xrightarrow[\psi]{} & G_1
\end{array}$$

commute, hence $\beta = \begin{pmatrix} \mathrm{id}_{X/\overline{\xi(G_0)}} & b \\ 0 & \mathrm{id}_{G_1} \end{pmatrix}$. The diagram

$$\begin{array}{ccc} G_0/nG_0 \oplus G_1[n] & \xrightarrow{(\frac{1}{n}\overline{\xi}, \iota_n^1)} & X/\overline{\xi(G_0)} \oplus G_1 \\ \Big\downarrow{\scriptstyle \alpha_n^0} & & \Big\downarrow{\scriptstyle \beta} \\ G_0/nG_0 \oplus G_1[n] & \xrightarrow{(\frac{1}{n}\overline{\xi}, \iota_n^1)} & X/\overline{\xi(G_0)} \oplus G_1 \end{array}$$

also commutes hence b must satisfy that the diagram

$$\begin{array}{ccc} G_1[n] & \hookrightarrow & G_1 \\ \Big\downarrow{\scriptstyle b_n^i} & & \Big\downarrow{\scriptstyle b} \\ G_0/nG_0 & \xrightarrow[\frac{1}{n}\overline{\xi}]{} & X/\overline{\xi(G_0)} \end{array}$$

commutes for $n \geq 2$. Hence $\ker(\widetilde{F}_{\underline{K}\mathcal{T}_u}) = \mathrm{Im}(\theta)$. By Corollary 7.24 there exists a splitting $\Phi : \mathrm{Aut}(K) \to \mathrm{Aut}(\mathbb{K})$, hence by Proposition 8.9 $\mathrm{Aut}(\mathbb{K}) \cong N(K) \rtimes \mathrm{Aut}(K)$ for the induced action. We show this action is Δ, from Example 8.8. Let $\left(\varphi, (\psi_n^i)_{i\in\{0,1\}, n\geq 2}\right) \in N(K)$ and $(\alpha^0, \alpha^1, \gamma) \in \mathrm{Aut}(K)$, we wish to show

$$\Phi(\alpha^0, \alpha^1, \gamma) \circ \theta\left(\varphi, (\psi_n^i)_{i\in\{0,1\}, n\geq 2}\right) \circ \Phi(\alpha^0, \alpha^1, \gamma)^{-1} = \theta\left(\Delta(\alpha^0, \alpha^1, \gamma)\left(\varphi, (\psi_n^i)_{i\in\{0,1\}, n\geq 2}\right)\right).$$

First we calculate the individual terms

$$\begin{aligned}
&\Phi(\alpha^0, \alpha^1, \gamma) = (\Lambda(\alpha^0, \alpha^1), \gamma, \overline{\upsilon} \circ \overline{\gamma} \circ \sigma_K + s_K \circ \alpha^1 \circ \psi) \\
&\Phi(\alpha^0, \alpha^1, \gamma)^{-1} = (\Lambda(\alpha^0, \alpha^1)^{-1}, \gamma^{-1}, \overline{\upsilon} \circ \overline{\gamma}^{-1} \circ \sigma_K + s_K \circ (\alpha^1)^{-1} \circ \psi) \\
&\theta\left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right) = \left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \begin{pmatrix} \mathrm{id}_{G_i/nG_i} & \psi_n^i \\ 0 & \mathrm{id}_{G_{1-i}[n]} \end{pmatrix}_{\substack{i\in\{0,1\},\\ n\geq 2}}\right), \mathrm{id}_X, \begin{pmatrix} \mathrm{id}_{X/\overline{\xi(G_0)}} & \varphi \\ 0 & \mathrm{id}_{G_1} \end{pmatrix}\right) \\
&\theta\left(\Delta(\alpha^0, \alpha^1, \gamma)\left(\varphi, (\psi_n^i)_{i\in\{0,1\}, n\geq 2}\right)\right) = \\
&\left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \begin{pmatrix} \mathrm{id}_{G_i/nG_i} & \overline{\alpha}_n^i \circ \psi_n^i \circ (\widetilde{\alpha}_n^{1-i})^{-1} \\ 0 & \mathrm{id}_{G_{1-i}[n]} \end{pmatrix}_{\substack{i\in\{0,1\},\\ n\geq 2}}\right), \mathrm{id}_X, \begin{pmatrix} \mathrm{id}_{X/\overline{\xi(G_0)}} & \overline{\gamma} \circ \varphi \circ (\alpha^1)^{-1} \\ 0 & \mathrm{id}_{G_1} \end{pmatrix}\right).
\end{aligned}$$

We observe

$$\begin{aligned}
\alpha_n^i \begin{pmatrix} [x]_n \\ y \end{pmatrix} &= (\overline{\mu}_n^i \circ \overline{\alpha}_n^i \circ \sigma_\Lambda + s_\Lambda \circ \widetilde{\alpha}_n^{1-i} \circ \overline{\nu}_n^i) \begin{pmatrix} [x]_n \\ y \end{pmatrix} = \begin{pmatrix} \overline{\alpha}([x]_n) \\ \widetilde{\alpha}_n^{1-i}(y) \end{pmatrix}, \\
\beta_\Phi \begin{pmatrix} [x]_{\overline{\xi(G_0)}} \\ y \end{pmatrix} &= (\overline{\upsilon} \circ \overline{\gamma} \circ \sigma_K + s_K \circ \alpha^1 \circ \psi) \begin{pmatrix} [x]_{\overline{\xi(G_0)}} \\ y \end{pmatrix} = \begin{pmatrix} \overline{\gamma}\left([x]_{\overline{\xi(G_0)}}\right) \\ \alpha^1(y) \end{pmatrix}.
\end{aligned}$$

Thus the matrix representations of $\alpha_n^i, (\alpha_n^i)^{-1}, \beta, \beta^{-1}$ becomes

$$\alpha_n^i = \begin{pmatrix} \overline{\alpha}_n^i & 0 \\ 0 & \widetilde{\alpha}_n^{1-i} \end{pmatrix},$$
$$(\alpha_n^i)^{-1} = \begin{pmatrix} (\overline{\alpha}_n^i)^{-1} & 0 \\ 0 & (\widetilde{\alpha}_n^{1-i})^{-1} \end{pmatrix},$$
$$\beta = \begin{pmatrix} \overline{\gamma} & 0 \\ 0 & \alpha^1 \end{pmatrix},$$
$$\beta^{-1} = \begin{pmatrix} (\overline{\gamma})^{-1} & 0 \\ 0 & (\alpha^1)^{-1} \end{pmatrix}.$$

For $i \in \{0,1\}$ and $n \geq 2$ using the above calculations

$$\Phi(\alpha^0, \alpha^1, \gamma) \circ \theta\left(\varphi, (\psi_n^i)_{i\in\{0,1\}, n\geq 2}\right) \circ \Phi(\alpha^0, \alpha^1, \gamma)^{-1} =$$
$$\Phi(\alpha^0, \alpha^1, \gamma) \circ \left(\left((\alpha^0)^{-1}, (\alpha^1)^{-1}, \begin{pmatrix} (\overline{\alpha}_n^i)^{-1} & \psi_n^i \circ (\widetilde{\alpha}_n^{1-i})^{-1} \\ 0 & (\widetilde{\alpha}_n^{1-i})^{-1} \end{pmatrix}\right), \gamma^{-1}, \begin{pmatrix} (\overline{\gamma})^{-1} & \varphi \circ (\alpha^1)^{-1} \\ 0 & (\alpha^1)^{-1} \end{pmatrix}\right) =$$
$$\left(\left(\mathrm{id}_{G_0}, \mathrm{id}_{G_1}, \begin{pmatrix} \mathrm{id}_{G_i/nG_i} & \overline{\alpha}_n^i \circ \psi_n^i \circ (\widetilde{\alpha}_n^{1-i})^{-1} \\ 0 & \mathrm{id}_{G_{1-i}[n]} \end{pmatrix}_{\substack{i\in\{0,1\},\\ n\geq 2}}\right), \mathrm{id}_X, \begin{pmatrix} \mathrm{id}_{X/\overline{\xi(G_0)}} & \overline{\gamma} \circ \varphi \circ (\alpha^1)^{-1} \\ 0 & \mathrm{id}_{G_1} \end{pmatrix}\right) =$$
$$\theta\left(\Delta(\alpha^0, \alpha^1, \gamma)\left(\varphi, (\psi_n^i)_{\substack{i\in\{0,1\},\\ n\geq 2}}\right)\right).$$

As the induced action from the splitting is Δ,

$$\mathrm{Aut}(\mathbb{K}) \cong N(K) \rtimes_\Delta \mathrm{Aut}(K).$$

□

We immediately apply the above in C^*-algebra land.

Corollary 8.11. *Let A be a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebra satisfying the universal coefficient theorem. Then*

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A) \cong N(KT_u(A)) \rtimes \mathrm{Aut}(KT_u(A)).$$

Proof. By Theorem 8.2 $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A) \cong \mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A))$ and by Theorem 8.10
$\mathrm{Aut}(\underline{\mathrm{K}}\mathrm{T_u}(A)) \cong N(KT_u(A)) \rtimes \mathrm{Aut}(KT_u(A))$. □

With this we end of the thesis, and the author sincerely hopes that the reader has enjoyed this small journey in C^*-algebra land, as well as the associated abstract nonsense.

9 Bibliography

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